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Quantifying virtual properties of Bianchi groups

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Quantifying virtual properties of Bianchi groups

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DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2018

In loving memory of Ernestina Berrío de Lasso.

Acknowledgments

I have been incredibly fortunate throughout my graduate journey and I am deeply grateful to so many people who have helped me along the way.

First and foremost, I would like to thank my advisor, Alan Reid, for his guidance, encouragement, and support. I am deeply indebted to Alan for all that he has taught me over the past several years. Thanks also to the former and current students (and grand-students) of Alan Reid. I am very lucky to be part of such a wonderful academic family.

I thank Cameron Gordon for officially adopting me when my advisor moved and for his time, support, and guidance. Thanks also to my committee members Daniel Allcock, Jeff Danciger, and Amir Mohammadi for numerous helpful conversations and advice for both research mathematics and my mathematics career. I thank Aaron Abrams, for introducing me to hyperbolic geometry and putting up with me as a undergraduate. Thanks also to many other mathematicians both at UT Austin and elsewhere who have provided encouragement, guidance, and friendship over the past several years.

I am grateful to many graduate students, postdocs, and staff members at the UT Math Department, as well as non-Math friends in Austin, without whose support and friendship I would not be who I am today. In particular I thank Han, Christine, Roberta, Kathryn, Cornelia, Soledad, Tim, Val, Luis, Allison, TR, Yen, Natalie, and many others. Thanks also to Karen Uhlenbeck and the ladies of DWM who continue to inspire me. Thanks to my friends back home for their continued friendship and support.

I am grateful to the Rice Mathematics department for hosting me during my final semester. Thanks in particular to the graduate students who welcomed me into their group.

None of this would be possible without the unconditional love and support of my family, especially my parents Alfredo and Tina, and my siblings Liza and David. Finally, I thank Andrew for always believing in me.

Quantifying virtual properties of Bianchi groups

by

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The University of Texas at Austin, 2018

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This dissertation is concerned with quantifying virtual properties of hyperbolic 3-manifold groups. We determine C-special subgroups of the Bianchi groups with index bounded above by 120 by effectivising the arguments of Agol-Long-Reid. These subgroups are congruence subgroups of small level and retract to the free group on two generators. As a consequence, we find a C-special 20-sheeted cover of the figure-eight knot complement. We also determine C-special congruence subgroups for a family of cocompact arithmetic Kleinian groups and a family of non-cocompact arithmetic groups of hyperbolic dimension 4.

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Chapter 1

Introduction

A 3-manifold M is said to have a *virtual* property if some finite-sheeted cover of M has that property. Similarly, a group Γ has a *virtual* property if some finite-index subgroup of Γ has that property. It was shown in [CLR97] that non-compact finite volume 3-manifolds virtually contain closed embedded essential surfaces. More recent progress in 3-manifold theory has determined the virtual Haken and the virtual fibering conjectures for all finite volume hyperbolic 3-manifolds, asserting that every finite volume hyperbolic 3-manifold virtually contains an embedded essential surface, and furthermore, is virtually a surface bundle over the circle. These results are implied by a stronger theorem which states that the fundamental groups of finite volume 3-manifolds are virtually special (see [Ago13] for the closed case and [Wis11] or [GM17] for the cusped case). A group is *special* as in [HW08] if it embeds in a right-angled Artin group (A-*special*) or in a right-angled Coxeter group (C-*special*). Every right-angled Artin group (RAAG) is a finite-index subgroup of a right-angled Coxeter group (RACG), so every special group embeds in a RACG [DJ00]. Virtually special groups inherit many nice properties from the RACG. In particular, for fundamental groups of hyperbolic 3-manifolds, virtually special (together with tameness [Ago04, CG06] and Canary's covering theorem

[Can96]) implies LERF and both the virtual Haken and the virtual fibering conjectures.

The goal of this dissertation is to address the following question:

Question 1.1. Given a virtually special group, can one determine a finite-index special subgroup? Can one bound its index?

Prior to the results of Agol and Wise [Ago13, Wis11], virtually special was known for several classes 3-manifold groups. Agol, Long, and Reid showed in [ALR01] that the Bianchi groups are virtually C-special. Later in [BHW11], Bergeron, Haglund, and Wise showed that the fundamental group of an arithmetic hyperbolic manifold of simplest type is virtually C-special. Recently, Question 1.1 was answered for the Seifert-Weber dodecahedral space, which is an arithmetic hyperbolic manifold of simplest type. In [ST17] Spreer and Tillmann constructed an A-special cover of the Seifert-Weber dodecahedral space with degree 60. In this paper we algebraically construct C-special covers of the Bianchi groups.

Theorem 3.1. *Let m be a square-free positive integer and $\mathcal{O}_m$ the ring of integers in the quadratic imaginary field $\mathbb{Q}(\sqrt{-m})$. The Bianchi group* $\mathrm{PSL}(2, \mathcal{O}_m)$ *contains a subgroup Δ_m which embeds as a geometrically finite subgroup in a RACG with index*

$$[\mathrm{PSL}(2, \mathcal{O}_m) : \Delta_m] = \begin{cases} 48 & \textit{if } m \equiv 1, 2 \mod (4) \\ 120 & \textit{if } m \equiv 3 \mod (8) \\ 72 & \textit{if } m \equiv 7 \mod (8) \end{cases}$$

where Δ_m is a principal congruence subgroup of level 2 if $m \equiv 1, 2 \mod (4)$ and is congruence of level 4 otherwise.

The proof of this result relies on making effective the strategy exploited by Agol, Long, and Reid in [ALR01]. The main idea is to realize quadratic forms associated to the Bianchi groups as sub-forms of the standard form of signature $(6, 1)$. The orthogonal group $O(6, 1; \mathbb{Z})$ contains the reflection group of a right-angled 6-dimensional hyperbolic Coxeter polytope. In fact, each $\mathbb{H}^3/\Delta_m$ immerses totally geodesically in this reflection orbifold.

A particular subgroup of a Bianchi group which has attracted much interest is the fundamental group of the figure-eight knot complement. In this case we can deduce the following corollary.

Corollary 3.11. *The figure-eight knot complement has a special finite-sheeted cover of degree 20.*

Another consequence of the embedding of Δ_m in a RACG is that Δ_m is virtually RFRS [Ago08] and virtually retracts to its geometrically finite subgroups [LR08].

Proposition 3.8. *The subgroup Δ_m retracts onto the free group on 2 generators and contains a RFRS subgroup of index at most 2^{26}.*

This dissertation is organized as follows: In Chapter 2 we give some preliminaries. We introduce Bianchi groups and their associated quadratic forms and prove Theorem 3.1 and Proposition 3.8 in Chapter 3. In Chapter 4 and Chapter 5 we describe two other families of arithmetic groups whose congruence subgroups of level 2 are special. The family in Chapter 4 is a family of cocompact Kleinian groups. The family in Chapter 5 is a family of rational arithmetic groups of hyperbolic dimension 4. The work in Chapter 3 and Chapter 4 will appear in [Chu17].

Chapter 2

Preliminaries

2.1 Quadratic forms

A reference for this section is [Cas08].

A *quadratic form* q over a number field K is a polynomial $q(t_1, \ldots, t_n)$ in $K[t_1, \ldots, t_n]$ such that each monomial term has degree 2, i.e.

$$q = \sum_{1 \leq i \leq j \leq n} a_{ij} t_i t_j$$

with $a_{ij} \in K$. Associated to the quadratic form q is a symmetric matrix S_q with entries a_{ij}. We say q is non-degenerate if $det(S_q) \neq 0$.

Two quadratic forms q and q' over n variables are *equivalent* (over K) if there exists $M \in \mathrm{GL}_n(K)$ such that $M^t S_q M = S_{q'}$.

A quadratic form q is *diagonal* if its symmetric matrix S_q is diagonal. The *signature* of a non-degenerate diagonal quadratic form q over $\mathbb{R}$ is (r, s) where r is the number of positive eigenvalues in S_q and s is the number of negative eigenvalues in S_q. Every quadratic form is equivalent to a diagonal form. We define the *signature* of a non-degenerate quadratic form q to be the signature of a diagonal quadratic form q' which is equivalent to q.

For any subgring R of $\mathbb{C}$, the orthogonal group $\mathrm{O}(q,R)$ of a quadratic form q in n variables is the group

$$\mathrm{O}(q,R) = \{M \in \mathrm{GL}(n,R) \mid M^t S_q M = S_q\}.$$

For K a real number field, we will also consider the orthogonal group $\mathrm{O}(q,\mathbb{R})$ defined as

$$\mathrm{O}(q,\mathbb{R}) = \{M \in \mathrm{GL}(n,\mathbb{R}) \mid M^t S_q M = S_q\}.$$

If q and q' are equivalent quadratic forms over a number field K, then $\mathrm{O}(q,K)$ and $\mathrm{O}(q',K)$ are isomorphic, and similarly if K is real then $\mathrm{O}(q,\mathbb{R})$ and $\mathrm{O}(q',\mathbb{R})$.

2.2 Hyperbolic space

The n-dimensional hyperbolic space $\mathbb{H}^n$ is the simply-connected Riemannian n-manifold with constant sectional curvature -1. There are several commonly studied models for hyperbolic space, each having its advantages and disadvantages. In what follows, we will focus on two particular models: the upper half-space model and the hyperboloid model. A good reference for this section is [BP92, Chapter A].

2.2.1 The upper half-space model

In the case of $\mathbb{H}^3$, the upper half-space model of hyperbolic space is particularly useful for understanding isometries via 2-by-2 matrices. Let

$$U^3 = \{(z,t) | z = x + iy \in \mathbb{C}, t \in \mathbb{R}_{>0}\}$$

and endow it with the complete Riemannian metric given by

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}.$$

The *upper half-space model* of hyperbolic 3-space $\mathbb{H}^3$ is identified with (U^3, ds) and has boundary $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The geodesics are the lines and semicircles in U^3 orthogonal to the plane $t = 0$.

Every linear fractional map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a transformation of the form

$$z \mapsto \frac{az + b}{cz + d},$$

and extends to an isometry of (U^3, ds) via the Poincaré extension. We may identify the group of orientation preserving isometries of $\mathbb{H}^3$ with $\mathrm{PSL}_2(\mathbb{C})$.

Isometries belonging to $\mathrm{PSL}_2(\mathbb{C})$ are classified into three categories distinguished by the fixed points of their action. Let γ be a non trivial element of $\mathrm{PSL}_2(\mathbb{C})$.

- γ is *elliptic* if the action of γ fixes a point in $\mathbb{H}^3$. In this case, $|\mathrm{tr}\gamma| < 2$.
- γ is *parabolic* if the action of γ has no fixed points in $\mathbb{H}^3$ but fixes one point in the boundary. In this case, $|\mathrm{tr}\gamma| = 2$.
- γ is *loxodromic* if the action of γ has no fixed points in $\mathbb{H}^3$ but fixes two points in the boundary. In this case, either $\mathrm{tr}\gamma$ is real and $|\mathrm{tr}\gamma| > 2$ or $\mathrm{tr}\gamma$ is not real. When γ is loxodromic and $\mathrm{tr}\gamma$ is real, γ is also known as *hyperbolic*.

2.2.2 The hyperboloid model

Let K a real number field and q a quadratic form over K in $n+1$ variables of signature $(n, 1)$. Let $\{x_0, x_1, \ldots, x_n\}$ be an orthogonal basis for f with $q(x_0) < 0$ and $q(x_i) > 0$ for each $i \geq 1$. The (column) vectors $v = (a_0, \ldots, a_n)^t$ with $q(v) = 1$ form an $n-$dimensional hyperboloid $\mathcal{C}$ consisting of a positive sheet $\mathcal{C}^+ = \{v \in \mathcal{C} | a_0 > 0\}$ and a negative sheet $\mathcal{C}^- = \{v \in \mathcal{C} | a_0 < 0\}$. The *hyperboloid model* of hyperbolic space $\mathbb{H}^n$ is identified with C^+.

The orthogonal group $\mathrm{O}(q, \mathbb{R})$ is the isometry group which preserves $\mathcal{C}$. The index two subgroup which preserves the positive sheet $\mathcal{C}^+$ is called $\mathrm{O}^+(q, \mathbb{R})$ (i.e. preserves the sign of a_0). This subgroup $\mathrm{O}^+(q, \mathbb{R})$ is identified with the full group of isometries of $\mathbb{H}^n$ and has two connected components, whose elements either preserve or reverse orientation. The orientation preserving isometry group of $\mathbb{H}^n$ is identified with the component of matrices with determinant 1, $\mathrm{SO}^+(q, \mathbb{R})$.

Note that when q has signature $(3, 1)$, $\mathrm{SO}^+(q, \mathbb{R})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$.

2.3 Discrete subgroups hyperbolic isometries

Good references for this section are [Bea95] and [BP92].

A discrete subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}^n)$ acts properly and discontinuosly on $\mathbb{H}^n$. Alternatively, any subgroup of of $\mathrm{Isom}^+(\mathbb{H}^n)$ acting properly and discontinuosly on $\mathbb{H}^n$ is discrete in $\mathrm{Isom}^+(\mathbb{H}^n)$.

Let $M_\Gamma = \mathbb{H}^n/\Gamma$ denote the quotient of $\mathbb{H}^n$ by a discrete group Γ with its induced metric. If Γ is torsion-free, then M_Γ is a *hyperbolic manifold* and $\pi_1(M_\Gamma) \cong \Gamma$. Otherwise M_Γ is a *hyperbolic orbifold*.

We say that Γ has *finite covolume* if M_Γ has finite volume. Such a Γ is known to be finitely presented. We call a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ a *lattice* if it has finite covolume. Every finite volume hyperbolic orbifold M_Γ has a thick-thin decomposition with a thick part consisting of a compact hyperbolic manifold possibly with boundary, and a thin part consisting of ends diffeomorphic to a flat $(n-1)$-manifold times a ray (see [BP92, Chapter D]). These thin ends are called *cusps*. Since every finite volume M_Γ is either closed or cusped, we call the corresponding Γ *cocompact* or *non-cocompact* respectively.

In dimension at least 3, we can completely understand a hyperbolic manifold by understanding its fundamental group. In fact, the topology and geometry of a finite-volume hyperbolic manifold is uniquely determined by its fundamental group. This was proved by Mostow in [Mos68] for closed hyperbolic manifolds and later generalized by Prasad in [Pra73].

Theorem 2.1 (Mostow-Prasad rigidity)**.** *Let M_{Γ_1} and M_{Γ_2} be complete finite-volume hyperbolic manifolds of dimension $n \geq 3$. Then M_{Γ_1} and M_{Γ_2} are homeomorphic if and only if they are isometric if and only if the discrete groups Γ_1 and Γ_2 are conjugate in* $\mathrm{Isom}(\mathbb{H}^n)$.

A discrete group Γ in $\mathrm{Isom}(\mathbb{H}^n)$ is called *geometrically finite* if M_Γ

contains a compact convex submanifold (its *convex core*) onto which it retracts.

2.3.1 Kleinian groups

A discrete subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}^3)$ is called a *Kleinian group*.

If Γ is a Kleinian group in $\mathrm{PSL}_2(\mathbb{C})$ and M_Γ is a finite-volume cusped hyperbolic 3-manifold, a cusp is isometric to a set of the form B/Γ_P where

$$B = \{(z,t) \in U^3 | t > 1\}$$

and $\Gamma_P \cong \mathbb{Z} \oplus \mathbb{Z}$ is a subgroup of Γ consisting entirely of parabolic elements stabilizing B. Topologically, B/Γ_P is homeomorphic to a torus times a ray.

2.4 Virtual properties

A group has a *virtual* property if there is a finite-index subgroup having that property. We say a 3-manifold has has a *virtual* property if there is a finite-sheeted cover of it having that property.

We say two subgroups H_1 and H_2 in G are *commensurable* if $H_1 \cap H_2$ has finite index in both H_1 and H_2, and *(weakly) commensurable* if some conjugate of H_1 is commensurable with H_2.

We say two orbifolds M_1 and M_2 are *commensurable* if there share a common finite-sheeted cover. Commensurability of complete finite-volume hyperbolic orbifolds $M_1 = \mathbb{H}^n/\Gamma_1$ and $M_2 \mathbb{H}^n/\Gamma_2$ is equivalent to (weak) commensurability of Γ_1 and Γ_2.

2.4.1 Virtual properties of groups

Let G be a group. We say a subgroup $H < G$ is *separable* if for every $g \in G \setminus H$, there exists a subgroup $K < G$ such that K has finite-index in G, K contains H, and $g \notin K$.

We say G is *residually finite*, or RF, if the identity subgroup is separable. The RF property is preserved by subgroups. We say G is *locally extended residually finite*, or LERF, if every finitely generated subgroup of G is separable.

Let G be a group and fix a generating set S. We say that H is a *quasi-convex subgroup* of G if there exists a $k \geq 0$ such that any geodesic in the Cayley graph $\mathrm{Cay}(G,S)$ having endpoints in H (viewed as a subset of $\mathrm{Cay}(G,S)$) is contained within the k-neighborhood of H.

We say G is *quasi-convex extended residually finite* (resp. *geometrically finite extended residually finite*), or QCERF (resp. GFERF), if every quasi-convex (resp. geometrically finite) subgroup is separable. The LERF, QCERF, and GFERF properties are preserved by appropriate subgroups, that is by finitely generated, quasi-convex, or geometrically finite subgroups. In the setting of lattices in $\mathrm{Isom}^+(\mathbb{H}^n)$, geometrically finite and quasi-convex are equivalent [Hru10, Corollary 1.3].

The rational derived series of a group G is defined inductively. If $G^{(1)}$ is the commutator subgroup $[G,G]$, then $G_r^{(1)} = \{x \in G | \exists k \neq 0 \text{ with } x^k \in G^{(1)}\}$. We say G is *residually finite rationally solvable*, or RFRS, if there is a sequence of subgroups $G = G_0 > G_1 > \cdots > G_i > \cdots$ with $G_i \triangleleft G$ such that $\cap_i G_i = 1$, $[G : G_i] < \infty$ and $G_{i+1} \geq (G_i)_r^{(1)}$. The RFRS property is preserved by subgroups.

2.4.2 Virtual properties of 3-manifolds

The separability properties of Kleinian groups are deeply connected to virtual topological properties of hyperbolic 3-manifolds.

An irreducible closed 3-manifold M is *Haken* if it contains a closed surface S and an embedding $S \hookrightarrow M$ which is π_1-injective and not boundary-parallel. A 3-manifold is *fibered* if it is a surface bundle over the circle.

Waldhausen first asked whether compact irreducible 3-manifolds with infinite fundamental groups are virtually Haken. Thurston later asked whether finite-volume hyperbolic 3-manifolds are not only virtually Haken, but actually virtually fibered. These two questions were promoted to the Virtual Haken conjecture and the Virtual fibering conjecture, and were proven in [Ago13] and [Wis11] (see Section 2.6).

2.5 RACGs and RAAGs

A *polytope* in n-dimensional hyperbolic space $\mathbb{H}^n$ is a convex polyhedron with finitely many actual and ideal vertices which is the convex hull of its vertices. A polytope is a *Coxeter polytope* if the dihedral angle between any two adjacent sides is either 0 or $\frac{\pi}{k}$ for some integer $k \geq 2$. A Coxeter polytope is *right-angled* if all the dihedral angles are either 0 or $\frac{\pi}{2}$.

The group generated by the reflections in the sides of a Coxeter polytope is a hyperbolic Coxeter group and is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. The group generated by reflections in the sides of a right-angled polytope is a *(geometric) right-angled Coxeter group.*

A right-angled Coxeter group, denoted *RACG*, is more generally constructed from a graph. Given a finite graph Δ with vertex set V and edge set E, the corresponding RACG is the group

$$C(\Delta) = \langle v \in V | v^2 = 1, [v, w] = 1 \text{ if } (v, w) \in E \rangle.$$

Two important properties of RACGs is that they are QCERF [Hag08] and virtually RFRS [Ago08]. Geometric RACGs are GFERF.

Another class closely related class of groups are the right-angled Artin groups, denoted RAAGs. They are also constructed from finite graphs as follows. Given a finite graph Δ with vertex set V and edge set E, the corresponding RAAG is the group

$$A(\Delta) = \langle v \in V | [v, w] = 1 \text{ if } (v, w) \in E \rangle.$$

The class of RAAGs contains free groups–corresponding to edge-less graphs, free abelian groups–corresponding to complete graphs, and many other interesting types of groups.

RAAGs and RACGs are commensurable. Every RAAG is contained in a RACG, although with different corresponding graphs [DJ00]. The commutator subgroup of a RACG is contained with finite index in the RAAG with the same corresponding graph [Dro03].

2.6 Special cube complexes and special groups

A *cube complex* X is a finite-dimensional CW-complex in which each cell is a cube and such that the attaching maps are combinatorial isomorphisms.

If C is a k-cube in X identified with $[-1,1]^k$, a *hyperplane* of C is any intersect of C with one of the coordinate hyperplanes of $\mathbb{R}^k$. Hyperplanes are then glued together where they meet in adjacent cubes to get hyperplanes of X.

There are four pathologies of hyperplanes in cube complexes to avoid [HW08, Ch. 3]:

(1) self-intersecting hyperplane

(2) one-sided hyperplane

(3) self-osculating hyperplane

(4) a pair of inter-osculating hyperplanes

These four pathologies are pictured in Figure 2.1 (sections of the 1-skeleton of the cube complex are in black, and the hyperplanes are in yellow).

A cube complex X is called *special* if pathologies (1),(3),(4) do not occur in X. X is *A-special* if it is special and pathology (2) also does not occur. X is *C-special* if it is special and the 1-skeleton of X is a bipartite graph.

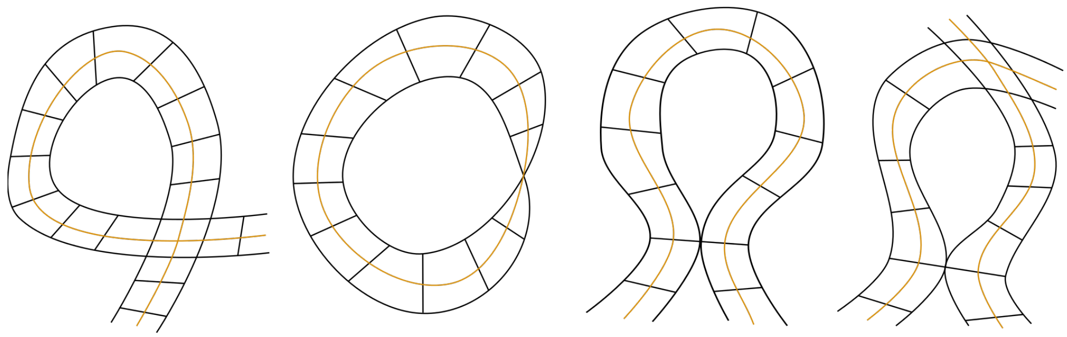

Figure 2.1: The four pathologies from left-to-right

A group is called *special* (resp. A-special, C-special) if it is the fundamental group of a special (resp. A-special, C-special) cube complex. A group is *compact special* if it is the fundamental group of a compact special cube complex.

Theorem 2.2 ([HW08]). *A group is compact A-special (resp. compact C-special) if and only if it embeds as a quasi-convex subgroup of a RAAG (resp. RACG).*

2.6.1 The virtually special theorem and its consequences

The following theorem was proved by Agol [Ago13] and Wise [Wis11], with major contributions from Agol-Groves-Manning [Ago13], Bergeron-Wise [BW12], Haglund-Wise [HW08, HW12], Hsu-Wise [HW15], Kahn-Markovic [KM12], and Sageev [Sag95, Sag97].

Theorem 2.3 (Virtually Special Theorem). *If M is a finite-volume hyperbolic 3-manifold, then $\pi_1(M)$ is virtually compact special.*

Virtually compact special groups inherit many nice properties from RACGs, which are known to be QCERF and virtually RFRS. However, in the case of finite covolume Kleinian groups, virtually special has stronger implications. The Tameness theorem of Agol [Ago04] and Calegari-Gabai [CG06] combined with Canary's covering theorem [Can96] gives a characterization of finitely generated subgroups of finite covolume Kleinian groups.

Theorem 2.4 (Subgroup Tameness Theorem)**.** *Let Γ be a finite covolume Kleinian group and let $H \leq \Gamma$ be a finitely generated subgroup. Then either H is a virtual surface fiber group or H is geometrically finite.*

The Virtually Special Theorem together with the Subgroup Tameness Theorem, work of Haglund-Wise [HW08], Agol [Ago08], Haglund [Hag08], and Long-Reid [LR08] implies the following corollary.

Corollary 2.5. *If M is a finite-volume hyperbolic 3-manifold, then*

- $\pi_1(M)$ *is LERF,*
- M *is virtually Haken,*
- M *is virtually fibered,*
- $\pi_1(M)$ *virtually retracts to its quasi-convex subgroups.*

2.7 Arithmetic groups

Good references for this section are [MR03] and [Mor15].

Arithmetic groups are lattices which arise as integral points of algebraic groups and were studied by Borel-Harish-Chandra [BHC62]. In the case of lattices in rank-1 Lie groups, and in particular lattices in $\mathrm{Isom}(\mathbb{H}^n)$, work of Margulis [Mar91] determined a definition in terms of the commensurator

$$\mathrm{Comm}_{\mathrm{Isom}(\mathbb{H}^n)}(\Gamma) = \{g \in \mathrm{Isom}(\mathbb{H}^n) | g\Gamma g^{-1}, \Gamma \text{ are commensurable}\}.$$

A lattice Γ in $\mathrm{Isom}(\mathbb{H}^n)$ is arithmetic if and only if its commensurator in $\mathrm{Isom}(\mathbb{H}^n)$ is dense. Alternatively, the commensurator of a non-arithmetic lattice is itself a lattice and is the unique maximal element in its commensurability class.

We now describe how to construct a type of arithmetic lattices via orthogonal groups of quadratic forms.

Let K is a totally real number field with ring of integers $\mathcal{O}_K$. Consider a quadratic form q of has signature $(n, 1)$ such that for every non-identity embedding $\sigma : k \hookrightarrow \mathbb{R}$, q^σ is positive-definite. Then the subgroups

$$\mathrm{O}^+(q, \mathcal{O}_K) = \mathrm{O}^+(q, \mathbb{R}) \cap \mathrm{GL}(n + 1, \mathcal{O}_K)$$

and

$$\mathrm{SO}^+(q, \mathcal{O}_K) = \mathrm{SO}^+(q, \mathbb{R}) \cap \mathrm{GL}(n + 1, \mathcal{O}_K)$$

is discrete in $\mathrm{O}^+(q, \mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^n)$ or $\mathrm{SO}^+(q, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^n)$.

Any discrete subgroup in $\mathrm{Isom}^+(\mathbb{H}^n)$ commensurable to some conjugate of $\mathrm{SO}^+(q, \mathcal{O}_K)$ is called *arithmetic of simplest type.* It follows from the classification of semisimple algebraic groups [Tit66] that in even dimensions all arithmetic groups in $\mathrm{Isom}^+(\mathbb{H}^n)$ are of simplest type. However, in odd dimensions, and in particular in dimension 3, there are other ways in which arithmetic groups may arise.

The quotient orbifolds of arithmetic lattices $\mathrm{SO}^+(q, \mathcal{O}_K)$ or those commensurable with them are cusped if and only if the quadratic form q is *isotropic*, i.e. there is a non-zero vector v such that $q(v) = 0$. Otherwise, the quotient orbifolds are compact.

Remark 2.6. Prior to the resolution of the virtually special theorem Theorem 2.3, virtually special was known for Bianchi groups, following from work of Agol-Long-Reid [ALR01], and more generally for arithmetic groups in $\mathrm{Isom}(\mathbb{H}^n)$ of simplest type, by Bergeron-Haglund-Wise [BHW11].

2.8 Congruence subgroups

A particularly interesting class of subgroups of arithmetic groups are the congruence subgroups. The existence of congruence subgroups in an arithmetic group provides it with a wealth of finite-index subgroups, and shows that the group is residually finite.

We define congruence subgroups for groups contained in some $\mathrm{SL}_n(\mathcal{O}_K)$. Note that the arithmetic groups $\mathrm{SO}^+(q, \mathcal{O}_K)$ are contained in $\mathrm{SL}_{n+1}(\mathcal{O}_K)$.

If $\mathscr{I}$ is an ideal in $\mathcal{O}_K$ and Γ is contained in $\mathrm{SL}_n(\mathcal{O}_K)$, the *principal congruence subgroup* of level $\mathscr{I}$ in Γ is the subgroup of matrices $M \in \Gamma$ which can be written as $M = I + \mathscr{I}A$ where $A \in \mathrm{GL}(n, R)$. We denote this subgroup by $\Gamma_{\mathscr{I}}$. Equivalently, $\Gamma_{(\mathscr{I})}$ is the kernel of the reduction homomorphism induced by the quotient map $\mathcal{O}_K \to \mathcal{O}_K/\mathscr{I}$.

The *norm* of $\mathscr{I}$ is given by $N(\mathscr{I}) = |\mathcal{O}_m/\mathscr{I}|$ and is multiplicative, that is $N(\mathscr{I}\mathscr{J}) = N(\mathscr{I})N(\mathscr{J})$.

If we decompose $\mathscr{I}$ as a product of powers of prime ideals $\mathscr{I} = \mathcal{P}_1^{j_1} \cdots \mathcal{P}_r^{j_r}$, as a consequence of the Chinese remainder theorem that, we have

$$\mathrm{SL}_n(\mathcal{O}_K/\mathscr{I}) = \mathrm{SL}_n(\mathcal{O}_K/\mathcal{P}_1^{j_1}) \times \cdots \times \mathrm{SL}_n(\mathcal{O}_K/\mathcal{P}_r^{j_r}).$$

Also, If $\mathscr{I}$ is a product of two ideals $\mathscr{I}_1, \mathscr{I}_2$ then $\Gamma_{\mathscr{I}_1\mathscr{I}_2} \subset \Gamma_{\mathscr{I}}$.

Any subgroup containing a principal congruence subgroup is called a *congruence subgroup*. Note that since the groups $\mathrm{SL}_n(\mathcal{O}_K/\mathscr{I})$ have finite order, congruence subgroups are necessarily of finite index.

Chapter 3

Special subgroups of Bianchi groups

The main goal of this chapter is to prove the following theorem and some consequences.

Theorem 3.1. *Let m be a square-free positive integer and $\mathcal{O}_m$ the ring of integers in the quadratic imaginary field $\mathbb{Q}(\sqrt{-m})$. The Bianchi group $\mathrm{PSL}(2, \mathcal{O}_m)$ contains a subgroup Δ_m which embeds as a geometrically finite subgroup in a RACG and has index*

$$[\mathrm{PSL}(2, \mathcal{O}_m) : \Delta_m] = \begin{cases} 48 & \text{if } m \equiv 1, 2 \mod (4) \\ 120 & \text{if } m \equiv 3 \mod (8) \\ 72 & \text{if } m \equiv 7 \mod (8) \end{cases}$$

where Δ_m is a principal congruence subgroup of level 2 if $m \equiv 1, 2 \mod (4)$ and is otherwise congruence of level 4.

3.1 Preliminaries

The Bianchi groups consists of the family of arithmetic Kleinian groups $\mathrm{PSL}(2, \mathcal{O}_m)$ where m is a positive square-free integer and

$$\mathcal{O}_m = \begin{cases} \mathbb{Z}[\sqrt{-m}] & m \equiv 1, 2 \mod (4) \\ \mathbb{Z}[\frac{1+\sqrt{-m}}{2}] & m \equiv 3 \mod (4) \end{cases} \tag{3.1}$$

is the ring of integers in the field $\mathbb{Q}(\sqrt{-m})$. Any non-cocompact arithmetic lattice in $\mathrm{Isom}^+(\mathbb{H}^3)$ is commensurable to some Bianchi group (see e.g. [MR03, §8.2]).

3.1.1 The Bianchi groups and quadratic forms

We now give a precise relationship between Bianchi groups and orthogonal groups of quadratic forms following [JM96, §2] and [EGM98, §1.3] (see Appendix A.1.3 for a correction to the computations in [EGM98, §1.3]). Computations for the maps below are included in Appendix A.1.

For m square-free integer, define the quadratic forms

$$Q_m = \begin{cases} 2x_0x_1 + 2x_2^2 + 2mx_3^2 & m \equiv 1,2 \mod 4 \\ 2x_0x_1 + 2x_2^2 + 2x_2x_3 + \frac{m+1}{2}x_3^2 & m \equiv 3 \mod 4 \end{cases} \tag{3.2}$$

with corresponding symmetric matrices

$$S_{m_{1,2}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2m \end{pmatrix} \text{ and } S_{m_3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & \frac{m+1}{2} \end{pmatrix}. \tag{3.3}$$

For $m \equiv 1,2 \mod 4$, define the homomorphism

$$\varphi_m : \mathrm{PGL}(2, \mathbb{Q}(\sqrt{-m})) \to \mathrm{SO}(Q_m, \mathbb{Q})$$

by sending $\alpha = \begin{pmatrix} a_0 + a_1\sqrt{-m} & b_0 + b_1\sqrt{-m} \\ c_0 + c_1\sqrt{-m} & d_0 + d_1\sqrt{-m} \end{pmatrix}$ to $(\det(\alpha))^{-1}$ times

$$\begin{pmatrix} d_0^2 + md_1^2 & -c_0^2 - mc_1^2 & 2\left(c_0d_0 + mc_1d_1\right) & -2m\left(c_1d_0 - c_0d_1\right) \\ -b_0^2 - mb_1^2 & a_0^2 + ma_1^2 & -2\left(a_0b_0 + ma_1b_1\right) & 2m\left(a_1b_0 - a_0b_1\right) \\ b_0d_0 + mb_1d_1 & -a_0c_0 - ma_1c_1 & b_0c_0 + mb_1c_1 + a_0d_0 + ma_1d_1 & m\left(b_1c_0 - b_0c_1 - a_1d_0 + a_0d_1\right) \\ b_1d_0 - b_0d_1 & a_0c_1 - a_1c_0 & b_1c_0 - b_0c_1 + a_1d_0 - a_0d_1 & -b_0c_0 - mb_1c_1 + a_0d_0 + ma_1d_1 \end{pmatrix}. \tag{3.4}$$

This homomorphism restricts to an injection

$$\mathrm{PSL}(2, \mathcal{O}_m) \hookrightarrow \mathrm{SO}^+(Q_m, \mathbb{Z})$$

whose image is the spinorial group [JM96].

For $m \equiv 3 \mod 4$, let $m = 4k - 1$ and define the homomorphism

$$\varphi_m : \mathrm{PGL}(2, \mathbb{Q}(\sqrt{-m})) \to \mathrm{SO}(Q_m, \mathbb{Q})$$

by sending $\alpha = \begin{pmatrix} a_0 + a_1\frac{1+\sqrt{-m}}{2} & b_0 + b_1\frac{1+\sqrt{-m}}{2} \\ c_0 + c_1\frac{1+\sqrt{-m}}{2} & d_0 + d_1\frac{1+\sqrt{-m}}{2} \end{pmatrix}$ to $(\det(\alpha))^{-1}$ times

$$\begin{pmatrix} d_0^2 + d_1d_0 + kd_1^2 & -c_0^2 - c_1c_0 - kc_1^2 & 2c_0d_0 + c_1d_0 + c_0d_1 + 2kc_1d_1 & c_0d_0 + 2kc_1d_0 - 2kc_0d_1 + c_0d_1 + kc_1d_1 \\ -b_0^2 - b_1b_0 - kb_1^2 & a_0^2 + a_1a_0 + ka_1^2 & -2a_0b_0 - a_1b_0 - a_0b_1 - 2ka_1b_1 & -a_0b_0 - 2ka_1b_0 + 2ka_0b_1 - a_0b_1 - ka_1b_1 \\ b_0d_0 + b_1d_0 + kb_1d_1 & -a_0c_0 - a_1c_0 - ka_1c_1 & b_0c_0 + b_1c_0 + kb_1c_1 + a_0d_0 + a_1d_0 + ka_1d_1 & b_0c_0 - kb_1c_0 + b_1c_0 + kb_0c_1 + kb_1c_1 + ka_1d_0 - ka_0d_1 \\ b_0d_1 - b_1d_0 & a_1c_0 - a_0c_1 & -b_1c_0 + b_0c_1 - a_1d_0 + a_0d_1 & -b_0c_0 - b_1c_0 - kb_1c_1 + a_0d_0 + a_0d_1 + ka_1d_1 \end{pmatrix}. \tag{3.5}$$

This homomorphism restricts to an injection

$$\mathrm{PSL}(2, \mathcal{O}_m) \hookrightarrow \mathrm{SO}^+(Q_m, \mathbb{Z})$$

whose image is the spinorial group [JM96].

3.1.2 Congruence subgroups of Bianchi groups

There is a formula for the index of the principal congruence subgroup $\mathrm{PSL}(2,\mathcal{O}_m)_{\mathscr{I}}$ using a decomposition of $\mathscr{I}$ into a product of powers of prime ideals $\mathscr{I} = \mathcal{P}_1^{j_1}\cdots\mathcal{P}_r^{j_r}$ (see [Dic01]):

$$[\mathrm{PSL}(2,\mathcal{O}_m) : \mathrm{PSL}(2,\mathcal{O}_m)_{\mathscr{I}}] = \begin{cases} 6 & \text{if } N(\mathscr{I}) = 2 \\ N(\mathscr{I})^3 \prod_{\mathcal{P}|\mathscr{I}} \left(1 - \frac{1}{N(\mathcal{P})^2}\right) & \text{if } 2 \in \mathscr{I} \\ \frac{N(\mathscr{I})^3}{2} \prod_{\mathcal{P}|\mathscr{I}} \left(1 - \frac{1}{N(\mathcal{P})^2}\right) & \text{otherwise .} \end{cases}$$

For a rational prime p there are three possibilities for the decomposition of the ideal (p) in $\mathcal{O}_m$:

$$(p) = \begin{cases} \mathcal{P}^2 & p \text{ is ramified and } N(\mathcal{P}) = p \\ \mathcal{P} & p \text{ is inert and } N(\mathcal{P}) = p^2 \\ \mathcal{P}_1\mathcal{P}_2 & p \text{ is split and } N(\mathcal{P}_1) = N(\mathcal{P}_2) = p. \end{cases}$$

Therefore since 2 is ramified when $m \equiv 1,2 \mod 4$, is inert when $m \equiv 3 \mod 4$, and is split when $m \equiv 7 \mod 4$ we have

$$[\mathrm{PSL}(2,\mathcal{O}_m) : \mathrm{PSL}(2,\mathcal{O}_m)_{(2)}] = \begin{cases} 48 & \text{if } m \equiv 1,2 \mod 4 \\ 60 & \text{if } m \equiv 3 \mod 8 \\ 36 & \text{if } m \equiv 7 \mod 8. \end{cases} \tag{3.6}$$

3.1.3 A family of RACGs

Let F_n be the diagonal quadratic form

$$F_n := -x_0^2 + x_1^2 + \cdots + x_n^2. \tag{3.7}$$

Theorem 3.2 ([ERT12, Theorem 2.1])**.** *For $2 \leq n \leq 7$, $\mathrm{O}^+(F_n,\mathbb{Z})_{(2)}$ is a RACG. It is the reflection group of an all-right hyperbolic polytope of dimen-*

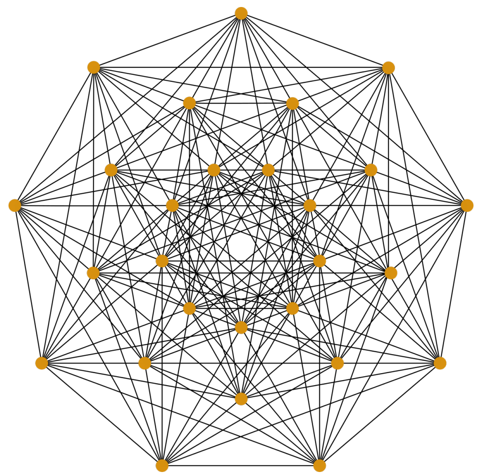

Figure 3.1: A projection of the 1-skeleton of the 6-dimensional Gosset polytope 2_{21} (adapted from [ERT12, Figure 4])

sion n, and $\mathrm{SO}^+(F_n,\mathbb{Z})_{(2)}$ *is its index 2 subgroup of orientation preserving isometries.*

Let P^n be the associated associated right-angled polyhedron with reflection group $\mathrm{O}^+(F_n,\mathbb{Z})_{(2)}$. For $n \leq 3 \leq 7$, each side of P^n is congruent to P^{n-1}. The polyhedron P^6 has 27 sides congruent to P^5 whose adjacency relations are represented by the 1-skeleton of the Gosset polytope 2_{21} pictured in Figure 3.1.

3.2 Proof of Theorem 3.1

3.2.1 The special subgroup for $m \equiv 1, 2 \mod 4$

By Equation 3.6 when $m \equiv 1, 2 \mod 4$ we have

$$\left[\mathrm{PSL}(2,\mathcal{O}_m) : \mathrm{PSL}(2,\mathcal{O}_m)_{(2)}\right] = 48. \tag{3.8}$$

Proposition 3.3. *Let $m \equiv 1, 2 \mod 4$ be square-free positive. The principal congruence subgroup $\mathrm{PSL}(2, \mathbb{Z}[\sqrt{-m}])_{(2)}$ embeds as a geometrically finite subgroup in the RACG $\mathrm{SO}^+(F, \mathbb{Z})_{(2)}$.*

Proof. Let $P_m := Q_m \oplus \langle 2m, 2m, 2m \rangle$. The group $\mathrm{SO}^+(Q_m, \mathbb{Z})$ is naturally a subgroup of $\mathrm{SO}^+(P_m, \mathbb{Z})$.

By Lagrange's 4-square theorem, write $m = w^2 + x^2 + y^2 + z^2$, a sum of four squares. Consider the 7×7 matrices

$$A_m = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & -x & -y & -z \\ 0 & 0 & 0 & x & w & z & -y \\ 0 & 0 & 0 & y & -z & w & x \\ 0 & 0 & 0 & z & y & -x & w \end{pmatrix} \tag{3.9}$$

with inverse

$$A_m^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{w}{m} & \frac{x}{m} & \frac{y}{m} & \frac{z}{m} \\ 0 & 0 & 0 & -\frac{x}{m} & \frac{w}{m} & -\frac{z}{m} & \frac{y}{m} \\ 0 & 0 & 0 & -\frac{y}{m} & \frac{z}{m} & \frac{w}{m} & -\frac{x}{m} \\ 0 & 0 & 0 & -\frac{z}{m} & -\frac{y}{m} & \frac{x}{m} & \frac{w}{m} \end{pmatrix}. \tag{3.10}$$

Let S_F be the diagonal matrix associated to F_6 and S_{P_m} the symmetric matrix associated to $\frac{1}{2}P_m$:

$$S_F = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

$$S_{P_m} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m \end{pmatrix}. \tag{3.12}$$

Then $A_m^t S_F A_m = S_{P_m}$. Since A_m has determinant $-m^2/2$, it is in $\mathrm{GL}(7, \mathbb{Q})$ and the forms F_6 and P_m are equivalent over $\mathbb{Q}$ and thus $A_m \mathrm{SO}^+(P_m, \mathbb{Q}) A_m^{-1} = \mathrm{SO}^+(F_6, \mathbb{Q})$. Therefore, $A_m \mathrm{SO}^+(Q_m, \mathbb{Q}) A_m^{-1} \subset \mathrm{SO}^+(F_6, \mathbb{Q})$.

A matrix N in $\mathrm{PSL}(2, \mathcal{O}_m)_{(2)}$ has form

$$\begin{pmatrix} 2a_0 + 1 + 2a_1\sqrt{-m} & 2b_0 + 2b_1\sqrt{-m} \\ 2c_0 + 2c_1\sqrt{-m} & 2d_0 + 1 + 2d_1\sqrt{-m} \end{pmatrix},$$

and from Equation 3.4, its image in φ_m is given by

$$\left(\begin{matrix} 4\left(d_0^2 + d_0 + md_1^2\right) + 1 & -4\left(c_0^2 + mc_1^2\right) \\ -4\left(b_0^2 + mb_1^2\right) & 4\left(a_0^2 + a_0 + ma_1^2\right) + 1 \\ 2\left(2d_0b_0 + b_0 + 2mb_1d_1\right) & -2\left(2a_0c_0 + c_0 + 2ma_1c_1\right) \\ 2\left(2d_0b_1 + b_1 - 2b_0d_1\right) & -2\left(2a_1c_0 - 2a_0c_1 - c_1\right) \end{matrix}\right.$$
$$\begin{matrix} 4\left(2d_0c_0 + c_0 + 2mc_1d_1\right) \\ -4\left(2a_0b_0 + b_0 + 2ma_1b_1\right) \\ 2\left(2d_0a_0 + a_0 + 2b_0c_0 + 2mb_1c_1 + d_0 + 2ma_1d_1\right) + 1 \\ 2\left(2d_0a_1 + a_1 + 2b_1c_0 - 2b_0c_1 - 2a_0d_1 - d_1\right) \end{matrix}$$
$$\left.\begin{matrix} -4m\left(2d_0c_1 + c_1 - 2c_0d_1\right) \\ 4m\left(2a_1b_0 - 2a_0b_1 - b_1\right) \\ 2m\left(-2d_0a_1 - a_1 + 2b_1c_0 - 2b_0c_1 + 2a_0d_1 + d_1\right) \\ 1 - 2\left(-2d_0a_0 - a_0 + 2b_0c_0 + 2mb_1c_1 - d_0 - 2ma_1d_1\right) \end{matrix}\right). \tag{3.13}$$

Let N' be

$$\begin{pmatrix} \varphi_m(N) & 0_{4\times 3} \\ 0_{3\times 4} & I_{3\times 3} \end{pmatrix} \in \mathrm{SO}^+(P_m, \mathbb{Z}).$$

It is now an easy check to see that $A_m N' A_m^{-1} \in \mathrm{SO}^+(F_6, \mathbb{Z})$ and furthermore $A_m N' A_m^{-1} \equiv I \mod (2)$ (see Appendix A.2). So $A_m N' A_m^{-1} \in \mathrm{SO}^+(F, \mathbb{Z})_{(2)}$.

□

Remark 3.4. From Equation 3.13 we see that

$$\varphi_m \left(\mathrm{PSL}(2, \mathcal{O}_m)\right)_{(4)} \leq \varphi_m \left(\mathrm{PSL}(2, \mathcal{O}_m)_{(2)}\right) \leq \varphi_m \left(\mathrm{PSL}(2, \mathcal{O}_m)\right)_{(2)}.$$

Remark 3.5. In the ring of integers $\mathcal{O}_m$ for the number field $\mathbb{Q}(\sqrt{-m})$, (2) is not always a prime ideal. One might ask whether a congruence subgroup with level a prime over 2 suffices to embed in a RACG. Unfortunately, using the methods above, a prime over 2 is not enough. Consider as an example the case of $m = 1$. The ideal (2) ramifies as $(1+i)^2$. However, the principal congruence subgroup $\mathrm{PSL}(2, \mathbb{Z}[i])_{(1+i)}$ is not contained in $\mathrm{SO}^+(F, \mathbb{Z})_{(2)}$ via the map φ_1 and the conjugation by $A_{m_{1,2}}$. Indeed if we write $1 = 1^2 + 0 + 0 + 0$ (i.e. $w = 1$, $x = y = z = 0$ in the definition of $A_{m_{1,2}}$ in Equation 3.9), the element

$$A_1^{-1} \varphi_1 \left(\begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix} \right) A_1 = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

does not reduce to the identity modulo 2.

3.2.2 The special subgroup for $m \equiv 3 \mod 4$

Proposition 3.6. *Let $m \equiv 3 \mod 4$ be square-free positive. The principal congruence subgroup $\mathrm{PSL}(2, \mathbb{Z}[\sqrt{-m}])_{(2)}$ has an index 2 subgroup Δ_m which* *embeds as a geometrically finite subgroup in the RACG* $\mathrm{SO}^+(F,\mathbb{Z})_{(2)}$ *and with*

$$[\mathrm{PSL}(2,\mathcal{O}_m):\Delta_m] = \begin{cases} 120 & \textit{if } m \equiv 3 \mod 8 \\ 72 & \textit{if } m \equiv 7 \mod 8. \end{cases} \tag{3.14}$$

Proof. Let $P_m := Q_m \oplus \langle 2m, 2m, 2m\rangle$. The group $\mathrm{SO}^+(Q_m,\mathbb{Z})$ is naturally a subgroup of $\mathrm{SO}^+(P_m,\mathbb{Z})$.

By Lagrange's 4-square theorem, write $m = w^2+x^2+y^2+z^2$, a sum of four squares. Consider the 7×7 matrices

$$A_{m_3} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{w}{2} & -x & -y & -z \\ 0 & 0 & 0 & -\frac{x}{2} & w & z & -y \\ 0 & 0 & 0 & -\frac{y}{2} & -z & w & x \\ 0 & 0 & 0 & -\frac{z}{2} & y & -x & w \end{pmatrix} \tag{3.15}$$

with inverse

$$A_{m_3}^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{w}{m} & \frac{x}{m} & \frac{y}{m} & \frac{z}{m} \\ 0 & 0 & 0 & -\frac{2w}{m} & -\frac{2x}{m} & -\frac{2y}{m} & -\frac{2z}{m} \\ 0 & 0 & 0 & -\frac{x}{m} & \frac{w}{m} & -\frac{z}{m} & \frac{y}{m} \\ 0 & 0 & 0 & -\frac{y}{m} & \frac{z}{m} & \frac{w}{m} & -\frac{x}{m} \\ 0 & 0 & 0 & -\frac{z}{m} & -\frac{y}{m} & \frac{x}{m} & \frac{w}{m} \end{pmatrix}. \tag{3.16}$$

Let S_F be the diagonal matrix associated to F_6 (see Equation 3.11) and S_{P_m} the symmetric matrix associated to $\frac{1}{2}P_m$:

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{m+1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m \end{pmatrix}. \tag{3.17}$$

Then $A_m^t S_F A_m = S_{P_m}$. Since A_m has determinant $-m^2/4$, it is in $\mathrm{GL}(7,\mathbb{Q})$ and the forms F_6 and P_m are equivalent over $\mathbb{Q}$ and thus $A_m \mathrm{SO}^+(P_m,\mathbb{Q}) A_m^{-1} = \mathrm{SO}^+(F_6,\mathbb{Q})$. Therefore, $A_m \mathrm{SO}^+(Q_m,\mathbb{Q}) A_m^{-1} \subset \mathrm{SO}^+(F_6,\mathbb{Q})$.

A matrix N in $\mathrm{PSL}(2,\mathcal{O}_m)_{(2)}$ has form

$$\begin{pmatrix} 2a_0+1+2a_1\frac{1+\sqrt{-m}}{2} & 2b_0+2b_1\frac{1+\sqrt{-m}}{2} \\ 2c_0+2c_1\frac{1+\sqrt{-m}}{2} & 2d_0+1+2d_1\frac{1+\sqrt{-m}}{2} \end{pmatrix},$$

and from Equation 3.5, its image in φ_m is given by

$$\left(\begin{matrix} 2d_1+4\left(d_0^2+d_1d_0+d_0+kd_1^2\right)+1 & -4\left(c_0^2+c_1c_0+kc_1^2\right) \\ -4\left(b_0^2+b_1b_0+kb_1^2\right) & 2a_1+4\left(a_0^2+a_1a_0+a_0+ka_1^2\right)+1 \\ 2(b_0+b_1)+4\left(d_0b_0+b_1d_0+kb_1d_1\right) & -2c_0-4\left(a_0c_0+a_1c_0+ka_1c_1\right) \\ -2b_1-4\left(d_0b_1-b_0d_1\right) & -2c_1+4\left(a_1c_0-a_0c_1\right) \end{matrix}\right.$$
$$\begin{matrix} 2(c_1)+4\left(2d_0c_0+d_1c_0+c_0+c_1d_0+2kc_1d_1\right) \\ -2b_1-4\left(2a_0b_0+a_1b_0+b_0+a_0b_1+2ka_1b_1\right) \\ 2(a_0+a_1+d_0)+4\left(d_0a_0+b_0c_0+b_1c_0+kb_1c_1+a_1d_0+ka_1d_1\right)+1 \\ 2(d_1-a_1)-4\left(d_0a_1+b_1c_0-b_0c_1-a_0d_1\right) \end{matrix}$$
$$\left.\begin{matrix} 2c_0+4(d_0c_0-2kd_1c_0+d_1c_0+kc_1+2kc_1d_0+kc_1d_1) \\ -2(b_0+b_1)-4\left(a_0b_0+2ka_1b_0-kb_1-2ka_0b_1+a_0b_1+ka_1b_1\right) \\ 2(ka_1-kd_1)+4(b_0c_0+b_1c_0)+4k\left(d_0a_1-b_1c_0+b_0c_1+b_1c_1-a_0d_1\right) \\ 1+2(a_0+d_0+d_1)-4\left(-d_0a_0-d_1a_0+b_0c_0+b_1c_0+kb_1c_1-ka_1d_1\right) \end{matrix}\right). \tag{3.18}$$

Since N has determinant 1, this implies

$$\begin{aligned} 1+2a_0+a_1-4b_0c_0-2b_1c_0-2b_0c_1-2b_1c_1+2d_0+4a_0d_0+2a_1d_0 \\ +d_1+2a_0d_1+4b_1c_1k-4a_1d_1k=1 \end{aligned} \tag{3.19}$$

$$\text{and } a_1-2b_1c_0-2b_0c_1-2b_1c_1+2a_1d_0+d_1+2a_0d_1+2a_1d_1=0$$

so $a_1 \equiv d_1 \mod 2$.

Let N' be

$$\begin{pmatrix} \varphi_m(N) & 0_{4\times 3} \\ 0_{3\times 4} & I_{3\times 3} \end{pmatrix} \in \mathrm{SO}^+(P_m, \mathbb{Z}).$$

It is now also true as in the case of $m \equiv 1, 2 \mod 4$ in Proposition 3.3 that $A_m N' A_m^{-1} \in \mathrm{SO}^+(F_6, \mathbb{Q})$ (see Appendix A.2).

Unfortunately, it is not always the case that $A_m N' A_m^{-1} \equiv I \mod (2)$, indeed

$$A_m N' A_m^{-1} \equiv \begin{pmatrix} 1 & 0 & \beta & w\beta & x\beta & y\beta & z\beta \\ 0 & 1 & \beta & w\beta & x\beta & y\beta & z\beta \\ \beta & \beta & 1 & 0 & 0 & 0 & 0 \\ w\beta & w\beta & 0 & 1 & 0 & 0 & 0 \\ x\beta & x\beta & 0 & 0 & 1 & 0 & 0 \\ y\beta & y\beta & 0 & 0 & 0 & 1 & 0 \\ z\beta & z\beta & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mod 2 \tag{3.20}$$

where $\beta = b_1 + c_1$. However, consider the subgroup of $\mathrm{PSL}(2, \mathcal{O}_m)_{(2)}$ of index 2 given by

$$\Delta_m := \left\{ \begin{pmatrix} 1 + 2(a_0 + a_1 \frac{1+\sqrt{-m}}{2}) & 2(b_0 + b_1 \frac{1+\sqrt{-m}}{2}) \\ 2(c_0 + c_1 \frac{1+\sqrt{-m}}{2}) & 1 + 2(d_0 + d_1 \frac{1+\sqrt{-m}}{2}) \end{pmatrix} : b_1 \equiv c_1 \mod 2 \right\}. \tag{3.21}$$

Then for $N \in \Delta_m$ $A_{m_i} N' A_{m_i}^{-1} \in \mathrm{SO}^+(F, \mathbb{Z})_{(2)}$. The proposition follows from Equation 3.6. $\square$

Remark 3.7. From Equation 3.18 and Equation 3.21 we see that

$$\varphi_m\left(\mathrm{PSL}(2, \mathcal{O}_m)\right)_{(4)} \leq \varphi_m\left(\mathrm{PSL}(2, \mathcal{O}_m)_{(2)}\right) \leq \varphi_m\left(\mathrm{PSL}(2, \mathcal{O}_m)\right)_{(2)}$$

and

$$\mathrm{PSL}(2, \mathcal{O}_m)_{(4)} \leq \Delta_m \leq \mathrm{PSL}(2, \mathcal{O}_m)_{(2)}$$

so Δ_m is congruence of level 4, but not principal congruence.

3.3 Virtual retracts and the RFRS condition

Another consequence of the embedding of Δ_m in a RACG is that Δ_m is virtually RFRS [Ago08] and virtually retracts to its geometrically finite subgroups [LR08].

Proposition 3.8. *The subgroup Δ_m retracts onto the free group on 2 generators and contains a RFRS subgroup of index at most 2^{26}.*

3.3.1 Virtual retractions

Let C be a hyperbolic right-angled Coxeter polytope of dimension $n > 2$, say in $\mathbb{H}^n$. Let Γ be the group generated by reflections on the faces of C. Define the graph Δ with a vertex for each face of C and an edge between two vertices if they meet at a right angle, so that $\Gamma \cong C(\Delta)$.

Let A be a subset of $V(\Delta)$ with induced subgraph Δ_A and B the complement of A in $V(\Delta)$ with induced subgraph Δ_B. Let Γ_A be the subgroup of Γ generated by the elements of A, and similarly define Γ_B. Let $N(\Gamma_B)$ denote the normal closure of Γ_B in Γ.

Lemma 3.9. $\Gamma/N(\Gamma_B) = \Gamma_A$*, that is,* Γ_A *is both a subgroup and a quotient of* Γ*. In particular, if F is a hyperface of dimension $n-k \geq 2$ in C and Γ_F the reflection group of F in* $\mathrm{Isom}(\mathbb{H}^{n-k})$*, then Γ retracts onto Γ_F.*

Proof. Consider the group presentation of $\Gamma/N(\Gamma_B)$ given by

$$\langle r \in V(\Delta) : r^2 = 1, (rs)^2 = 1 \text{ if } (r,s) \in E(\Delta), t = 1 \text{ if } t \in V(\Delta_B)\rangle.$$

If $t \in V(\Delta_B)$, then for any $r \in V(\Delta)$, with $(r,t) \in E(\Delta)$, the relation $(rt)^2 = 1$ is equivalent in $\Gamma/N(\Gamma_B)$ to $r^2 = 1$, which we already had. Therefore,

$$\begin{aligned}\Gamma/N(\Gamma_B) &= \langle r \in V(\Delta) - V(\Delta_B) : r^2 = 1, (rs)^2 = 1 \text{ if } (r,s) \in E(\Delta) \rangle \\ &= \langle r \in V(\Delta_A) : r^2 = 1, (rs)^2 = 1 \text{ if } (r,s) \in E(\Delta_A) \rangle \\ &= \Gamma_A.\end{aligned}$$

From [AVS93] we have the following facts:

- Any face of a hyperbolic right-angled polytope of dimension $n > 2$ is a hyperbolic right-angled polytope of dimension $n-2$.
- Every k-dimensional face of a hyperbolic right-angled polytope belongs only to $n-k$ many hyperfaces.

In particular, any $(n-3)$-dimensional face (or vertex in the case $n = 3$) belongs only to 3 hyperfaces. This also means that any two $(n-1)$ faces which intersect, must intersect in an $(n-2)$ face.

Now, suppose that F is a face of C, i.e. a hyperplane of dimension $(n-1)$. Identify $H = \mathbb{H}^{n-1}$ with the hyperplane of $\mathbb{H}^n$ containing the face F. Let r be the reflection on H, and hence on F. Let F' be a face of C which meets F at a right-angle. Then r', reflection on F', stabilizes H. Indeed, restricted to H, r' is a reflection on the face of F where F' meets F.

Suppose F_1 and F_2 both meet F and intersect at an $(n-3)$-dimensional face of C. By the facts above, F_1 and F_2 intersect at an $(n-2)$-dimensional face at a right-angle. Therefore for the corresponding reflections r_1 and r_2, we have that $(r_1 r_2)^2 = 1$.

Let S be the set of faces of C which meet F in $\mathbb{H}^n$ and not including F itself. The map $\phi : \Gamma \to \Gamma_S$ given by

$$r_i \mapsto \begin{cases} r_i & \text{if } F_i \in S \\ 1 & \text{if } F_i \notin S \end{cases}$$

is a retraction.

For a lower-dimensional hyperface (of dimension at least 2), repeat this process and compose to get a retraction from Γ onto Γ_F. $\square$

3.3.2 The RFRS condition

In [Ago08] Agol introduced the RFRS condition as a criteria for 3-manifolds to virtually fiber. He showed the following theorem.

Theorem 3.10 ([Ago08, Theorem 2.2]). *Let G be a RACG. Its commutator subgroup $[G, G]$ is RFRS.*

The RFRS condition is preserved by subgroups and if the fundamental group of a 3-manifold is virtually RFRS, then the 3-manifold vritually fibers [Ago08]. Therefore, Theorem 3.10 together with Theorem 2.3 show that finite-volume hyperbolic 3-manifolds virtually fiber.

We also note here that the commutator subgroup of the RACG $C(\Delta)$ for some graph Δ is contained in the RAAG $A(\Delta)$ [Dro03].

3.3.3 Proof of Proposition 3.8

Proof of Proposition 3.8. Each of the 2-dimensional faces of the hyperbolic Coxeter polyhedron of dimension 6 with corresponding RACG $\mathrm{SO}^+(F_6,\mathbb{Z})_{(2)}$ is a 2-dimensional hyperbolic Coxeter polygon with RACG $\mathrm{SO}^+(F_2,\mathbb{Z})_{(2)}$ (refer to Theorem 3.2). The intersection of the image of Δ_m (after φ_m and conjugation by A_m) is exactly

$$\Delta_m \cap \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{PSL}(2,\mathbb{Z})_{(2)} = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

a free group on 2 generators.

This polyhedron with corresponding RACG $\mathrm{SO}^+(F_6,\mathbb{Z})_{(2)}$ has 27 sides. Therefore, the commutator subgroup of the reflection group $\mathrm{O}^+(F_6,\mathbb{Z})_{(2)}$ has index 2^{27} in $\mathrm{O}^+(F_6,\mathbb{Z})_{(2)}$ and index 2^{26} in $\mathrm{SO}^+(F_6,\mathbb{Z})_{(2)}$. The proposition now follows from Theorem 3.10. □

3.4 Examples of hyperbolic links complements

Some particular examples of hyperbolic link complements in $\mathbb{S}^3$ which have attracted much interest are those of the figure-eight knot, the Whitehead link, and the Borromean rings. Since these three have fundamental groups contained in Bianchi groups, Theorem 3.1 can be applied to determine special subgroups of these. However, we will see in some cases that the bounds from Theorem 3.1 might not be optimal.

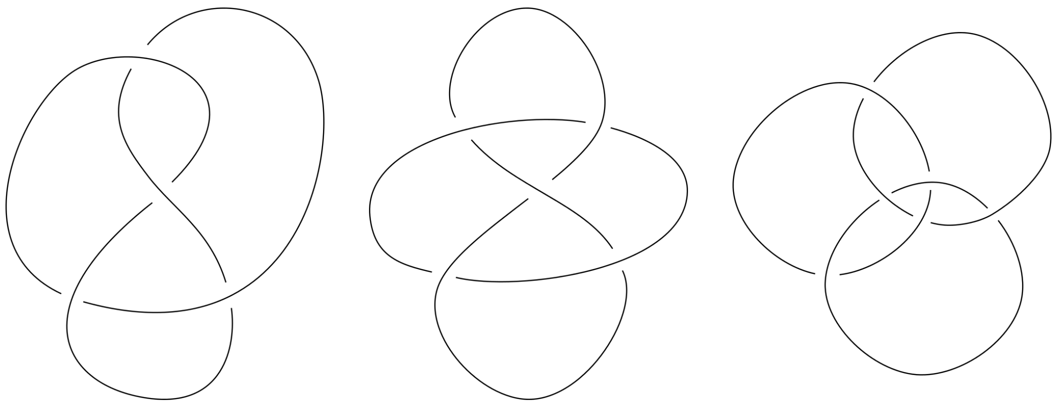

Figure 3.2: figure-eight knot, Whitehead link, and Borromean rings

3.4.1 The figure 8 knot

The figure-eight knot complement has fundamental group [Ril75]

$$\Gamma_8 := \langle x, y | (x^{-1}yxy^{-1})x = y(x^{-1}yxy^{-1})\rangle.$$

The holonomy representation of the hyperbolic structure of the figure-eight knot complement is given by

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$$

where $\omega = \frac{1+\sqrt{-3}}{2}$ [Ril75]. It determines a subgroup of the Bianchi group $\mathrm{PSL}(2, \mathcal{O}_3)$ of index 12. The intersection $\Gamma_8 \cap \Delta_3$ is then a special subgroup of Γ_8.

The π_2 reduction of Γ_8 induced by $\mathcal{O}_3 \to \mathcal{O}_3/(2)$ is the dihedral group D_5 with presentation $\langle a, b | a^2 = b^2 = (ab)^5 = 1\rangle$ where $a = \pi_2(x)$ and $b = \pi_2(y)$ (see Appendix B.1). Therefore, $\Gamma_8 \cap \mathrm{PSL}(2, \mathcal{O}_3)_{(2)}$ has index 10 in Γ_8. Note that the element

$$\begin{pmatrix} 1 & 0 \\ 2\omega & 1 \end{pmatrix}$$

is contained in $\Gamma_8 \cap \mathrm{PSL}(2, \mathcal{O}_3)_{(2)}$ but is not in Δ_3 (recall the definition of Δ_3 from Equation 3.21). We get the following corollary.

Corollary 3.11. *The figure-eight knot complement has a special finite-sheeted cover of degree 20.*

The special subgroup $\Gamma_8 \cap \Delta_3$ in Γ_8 of index 20 is generated by the following 8 matrices:

$$\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4\omega & 1 \end{pmatrix}, \begin{pmatrix} 1+2\omega & -2\omega \\ 2\omega & 1-2\omega \end{pmatrix}, \begin{pmatrix} 1-4\omega & 2 \\ 8-8\omega & 1+4\omega \end{pmatrix}, \right.$$
$$\begin{pmatrix} -1+4\omega & -6\omega \\ -2+2\omega & 3-4\omega \end{pmatrix}, \begin{pmatrix} -3+6\omega & -2\omega \\ -16+10\omega & 5-6\omega \end{pmatrix},$$
$$\left. \begin{pmatrix} -39+22\omega & 10-16\omega \\ -26+8\omega & 9-10\omega \end{pmatrix}, \begin{pmatrix} 41-22\omega & -10+16\omega \\ 26-12\omega & -7+10\omega \end{pmatrix} \right\}.$$

3.4.2 The Whitehead link

The Whitehead link complement has fundamental group [Wie78]

$$\Gamma_W := \langle x, y | [x,y][x,y^{-1}][x^{-1},y^{-1}][x^{-1},y] \rangle.$$

The holonomy representation of the hyperbolic structure of the Whitehead link complement is given by

$$x \mapsto \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ -1-i & 1 \end{pmatrix}$$

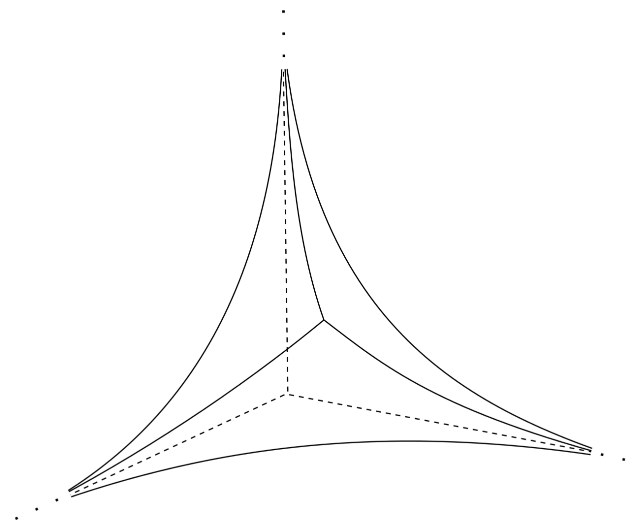

Figure 3.3: The polyhedron P^3 (adapted from [ERT12, Figure 2])

[Wie78, Example 1]. It determines a subgroup of the Bianchi group $\mathrm{PSL}(2, \mathcal{O}_1)$ of index 12. The intersection $\Gamma_W \cap \Delta_1$ is then a special subgroup of Γ_W with $[\Gamma_W : \Gamma_W \cap \Delta_1] = 8$.

However, in this case, this bound is not optimal. The Whitehead link complement is covered by the complement of the link 8^3_9, which is special. Ratcliffe-Tschantz showed that the fundamental group of the complement of 8^3_9 is realized by a minimal-index torsion free subgroup of $\mathrm{O}^+(F_3, \mathbb{Z})_{(2)}$ [RT00, §3]. In particular, it is contained in the reflection group for the polyhedron P^3 pictured in Figure 3.3, a hyperbolic non-compact right-angled 6-sided polyhedron with 2 actual vertices and 3 ideal vertices. The 8^3_9 link complement is covered by 8 copies of P^3.

The 8^3_9 link complement is the double cover of the Whitehead link complement branched along one component. Its fundamental group as a subgroup of Γ_W is generated by $\{y, x^2, xy^{-1}x^{-1}\}$ [BFLW84]. The holonomy representations of the hyperbolic structure of the 8^3_9 link complement is given by

$$y \mapsto \begin{pmatrix} 1 & 0 \\ -1-i & 1 \end{pmatrix}, x^2 \mapsto \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}, xy^{-1}x^{-1} \mapsto \begin{pmatrix} i & 1+i \\ 1+i & 2-i \end{pmatrix}.$$

Therefore, Γ_W has a special subgroup of index 2.

3.4.3 The Borromean rings

The Borromean rings complement has fundamental group [Wie78]

$$\Gamma_B := \langle x, y, z | (yzy^{-1}z^{-1})x(yzy^{-1}z^{-1})^{-1}x^{-1}, (z^{-1}xzx^{-1})y(z^{-1}xzx^{-1})^{-1}y^{-1}\rangle.$$

The holonomy representation of the hyperbolic structure of the Borromean rings complement is given by

$$x \mapsto \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}, y \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, z \mapsto \begin{pmatrix} 2+i & 2i \\ -1 & -i \end{pmatrix}$$

[Wie78, Example 8]. It determines a subgroup of the Bianchi group $\mathrm{PSL}(2, \mathcal{O}_1)$ of index 24. The intersection $\Gamma_B \cap \Delta_1$ is then a special subgroup of Γ_B with $[\Gamma_B : \Gamma_B \cap \Delta_1] = 4$.

As in the Whitehead link case, this bound is not optimal. In fact, the Borromean rings complement is itself special. Ratcliffe-Tschantz showed that Γ_B is also realized by a minimal-index torsion free subgroup of $\mathrm{O}^+(F_3, \mathbb{Z})_{(2)}$ and is also contained in the reflection group for P^3 with the Borromean rings complement covered by 8 copies of P^3 [RT00, §3].

Chapter 4

Special subgroups of a family of Cocompact Kleinian groups

In this chapter we first consider a family of cocompact Kleinian known to virtually embed in a RACG. We then focus on a particular group in the family and a Kleinian group in its commensurability class whose quotient closed hyperbolic 3-manifold is non-Haken.

4.1 A family of cocompact Kleinian groups

Choose a positive prime $m \equiv -1 \mod (8)$ and let Q'_m be the quadratic form

$$Q'_m := -mx_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{4.1}$$

By Dirichlet's Theorem there are infinitely many such primes. For different m, the groups $\mathrm{SO}^+(Q'_m, \mathbb{Z})$ are non-comensurable cocompact arithmetic lattices which all virtually embed in a RACG (see [ALR01, Lemma 4.6(1),§6]).

Let P'_m be the quadratic form $-mx_1^2 + x_2^2 + x_3^2 + x_4^2 + mx_5^2$. Then $P'_m = Q'_m \oplus \langle m \rangle$. The group $\mathrm{SO}^+(Q'_m, \mathbb{Z})$ is naturally a subgroup of $\mathrm{SO}^+(P'_m, \mathbb{Z})$.

Lemma 4.1. *The reduction modulo m of the first column of an element in $\mathrm{SO}^+(Q'_m, \mathbb{Z})$ is $(\pm 1, 0, 0, 0)$.*

Proof. Suppose $N' \in \mathrm{SO}^+(Q'_m, \mathbb{Z})$ and $N = \pi_m(N')$. Then N can be written as a block matrix

$$\begin{pmatrix} N_0 & N_1 \\ N_2 & N_3 \end{pmatrix}$$

where N_0 is a 1×1 matrix, N_1 is 1×3, N_2 is 3×1 and N_3 is 3×3. Since N must satisfy $N^t \pi_m(S_{Q'_m}) N = \pi_m(S_{Q'_m})$ we have

$$N^t \pi_m(S_{Q'_m}) N = \begin{pmatrix} N_0^t & N_2^t \\ N_1^t & N_3^t \end{pmatrix} \begin{pmatrix} 0_{1\times 1} & 0_{1\times 3} \\ 0_{3\times 1} & I_{3\times 3} \end{pmatrix} \begin{pmatrix} N_0 & N_1 \\ N_2 & N_3 \end{pmatrix} \tag{4.2}$$

$$= \begin{pmatrix} N_2^t N_2 & N_2^t N_3 \\ N_3^t N_2 & N_3^t N_3 \end{pmatrix} \tag{4.3}$$

$$\equiv \begin{pmatrix} 0_{1\times 1} & 0_{1\times 3} \\ 0_{3\times 1} & I_{3\times 3} \end{pmatrix}. \tag{4.4}$$

Since m is prime, $\mathbb{Z}/m\mathbb{Z}$ is a field and $N_3^t N_3 \equiv I_{3\times 3}$ together with $N_2^t N_3 \equiv 0_{1\times 3}$ implies $N_2 \equiv 0_{1\times 3}$. Therefore, the first column of N' has form $(a, bm, cm, dm)^t$ and must satisfy $-ma^2 + (mb)^2 + (mc)^2 + (md)^2 = -m$. Dividing by m, we get that $-a^2 + mb^2 + mc^2 + md^2 = -1$. This implies $a^2 = 1 \mod m$ and so $a = \pm 1 \mod m$ and the first column is $(\pm 1, 0, 0, 0)^t$ modulo m. □

Remark 4.2. An element $\mathrm{O}^+(Q'_m, \mathbb{Z})$ is in $\mathrm{SO}^+(Q'_m, \mathbb{Z})$ if the $(1,1)$-matrix entry is positive. Notice that the group $\mathrm{SO}^+(Q'_m, \mathbb{Z})$ contains elements whose first column reduces to $(-1, 0, 0, 0)^t$. A classical result in number theory known as the 3-squares-theorem states that a positive integer can be written as a sum of three squares if and only if it is not of the form $4(8b + 7)$. Therefore, $m - 2$

can be written as a sum of 3 squares. Let $m-2=x^2+y^2+z^2$, then

$$\begin{pmatrix} m-1 & x & y & -z \\ -xm & -x^2-1 & -xy & xz \\ -ym & -xy & -y^2-1 & yz \\ -zm & -xz & -yz & d^2+1 \end{pmatrix} \in \mathrm{SO}^+(Q'_m, \mathbb{Z}).$$

For example, when $m=7$, then $7-2=5=2^2+1^2$ and

$$\begin{pmatrix} 6 & 1 & 2 & 0 \\ -7 & -2 & -2 & 0 \\ -14 & -2 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}^+(Q'_7, \mathbb{Z}).$$

Let Δ^m be the index 2 subgroup of $\mathrm{SO}^+(Q'_m,\mathbb{Z})$ of elements whose $(1,1)$-matrix entry is equivalent to 1 modulo m.

Recall the quadratic form $F_4=-x_0^2+x_1^2+x_2^2+x_3^2+x_4^2$ from Equation 3.7.

Proposition 4.3. *Let $m=8k-1$ be a positive prime number. The group $\Delta^m_{(2)}$ embeds in the RACG $\mathrm{SO}^+(F_4,\mathbb{Z})_{(2)}$.*

Proof. Note first that $m=(4k)^2-(4k-1)^2$, a difference of two squares. Consider the 5×5 matrix

$$A'_m = \begin{pmatrix} 4k & 0 & 0 & 0 & -(4k-1) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -(4k-1) & 0 & 0 & 0 & 4k \end{pmatrix} \tag{4.5}$$

with inverse

$$(A'_m)^{-1} = \begin{pmatrix} \frac{4k}{m} & 0 & 0 & 0 & \frac{4k-1}{m} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{4k-1}{m} & 0 & 0 & 0 & \frac{4k}{m} \end{pmatrix}. \tag{4.6}$$

Let S_F be the diagonal matrix associated to F_4 and $S_{P'_m}$ the symmetric matrix associated to P'_m:

$$S_F = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S_{P'_m} = \begin{pmatrix} -m & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & m \end{pmatrix}. \tag{4.7}$$

Then $(A'_m)^t S_F A'_m = S_{P'_m}$. Since A'_m has determinant m, it is in $\mathrm{GL}(5, \mathbb{Q})$. Therefore, the forms F_4 and P'_m are equivalent over $\mathbb{Q}$ and thus

$$A'_m SO^+(P'_m, \mathbb{Q})(A'_m)^{-1} = SO^+(F_4, \mathbb{Q})$$

with $A'_m SO^+(P'_m, \mathbb{Q})(A'_m)^{-1}$ and $SO^+(F_4, \mathbb{Z})$ commensurable.

By Lemma 4.1, a matrix N in $\Delta^m_{(2)}$ sits naturally in $SO^+(P'_m, \mathbb{Z})$ with form

$$\begin{pmatrix} 2ma_1+1 & 2b_1 & 2c_1 & 2d_1 & 0 \\ 2ma_2 & 2b_2+1 & 2c_2 & 2d_2 & 0 \\ 2ma_3 & 2b_3 & 2c_3+1 & 2d_3 & 0 \\ 2ma_4 & 2b_4 & 2c_4 & 2d_4+1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A'_m N (A'_m)^{-1} = \begin{pmatrix} 32a_1k^2+1 & 8kb_1 & 8kc_1 & 8kd_1 & 32k^2a_1 - 8ka_1 \\ 8ka_2 & 2b_2+1 & 2c_2 & 2d_2 & 8ka_2 - 2a_2 \\ 8ka_3 & 2b_3 & 2c_3+1 & 2d_3 & 8ka_3 - 2a_3 \\ 8ka_4 & 2b_4 & 2c_4 & 2d_4+1 & 8ka_4 - 2a_4 \\ 8ka_1 - 32k^2a_1 & 2b_1 - 8kb_1 & 2c_1 - 8kc_1 & 2d_1 - 8kd_1 & 16a_1k - 32a_1k^2 - 2a_1 + 1 \end{pmatrix} \tag{4.8}$$

is in $\mathrm{SO}^+(F_4, \mathbb{Z})_{(2)}$ (see Appendix A.3). $\square$

4.2 A non-Haken example

Commensurable with $\mathrm{SO}^+(Q'_7, \mathbb{Z})$ and $\Delta^7_{(2)}$ is the fundamental group of a particular non-Haken hyperbolic 3-manifold mentioned in [ALR01, p.616]. For this reason, we treat the case of $\mathrm{SO}^+(Q'_7, \mathbb{Z})$ with more detail. The computations for this section are included in Appendix B.2.

The arithmetic hyperbolic tetrahedral groups were studied in [MR89]. The tetrahedron T_6 of [MR89] has dihedral angles $\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}; \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}\}$ and volume approximately 0.2222287320. Let Γ be the index-2 subgroup of orientation preserving isometries in the reflection group for this tetrahedron. It is an arithmetic Kleinian group generated by the matrices

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 9/2 & 1/2 & 3/2 & -1/2 \\ -7/2 & 1/2 & -3/2 & 1/2 \\ -21/2 & -3/2 & -7/1 & 3/2 \\ -7/2 & -1/2 & -3/2 & -1/2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\}.$$

Γ is commensurable with the group $\mathrm{SO}^+(Q'_7, \mathbb{Z})$ generated by the matrices

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 1 & 2 & 0 \\ -7 & -2 & -2 & 0 \\ -14 & -2 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

It turns out that $\Gamma \cap \mathrm{SO}^+(Q'_7, \mathbb{Z})$ is actually the group Δ^7. It has index

3 in Γ and index 2 in $\mathrm{SO}^+(Q_7', \mathbb{Z})$ and is generated by the matrices

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 8 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ -21 & 0 & -8 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 8 & 2 & 2 & 1 \\ -14 & -3 & -4 & 2 \\ -14 & -4 & -3 & 2 \\ -7 & -2 & -2 & 0 \end{pmatrix} \right\}.$$

Magma shows that the reduction of Δ^7 modulo 2 has order 24. Therefore $[\Delta^7 : \Delta^7_{(2)}] = 24$ and since $[\Gamma : \Delta^7] = 3$, then Γ has a special subgroup of index 72.

Both groups Γ and $\mathrm{SO}^+(Q_7', \mathbb{Z})$ are contained in the same maximal arithmetic Kleinian group, (an image of) $\Gamma_{\mathcal{O}}$ where $\mathcal{O}$ is a maximal order in the invariant quaternion algebra (with notation as in [MR89]) with volume approximately 0.1111143660. It is shown in [MR89] that $\Gamma^{(2)}$, the subgroup of Γ generated by all the squares, is the group $\Gamma^1_{\mathcal{O}} = P\rho_1(\mathcal{O}^1)$.

There is a non-Haken hyperbolic 3-manifold obtained by a 4/1-Dehn filling on the once-punctured torus bundle with monodromy R^2L^2 (given by the RL-factorization). It has volume ≈ 2.666744783. Via Snap, one can check that its fundamental group, call it Θ, is commensurable with Γ because they have the same invariant arithmetic data Appendix B.2. This group Θ is not contained in $\Gamma_{\mathcal{O}}$, but in some other maximal group. However, $\Theta^{(2)}$ is contained in $\Gamma^1_{\mathcal{O}}$ with index bounded above by 8. Therefore, Θ has a special subgroup of index bounded above by $8 \cdot 72 = 576$.

Chapter 5

Special subgroups of some arithmetic groups in dimension 4

In this chapter consider arithmetic lattices from orthogonal groups of integral quadratic forms of signature $(4,1)$. Their quotients determine cusped arithmetic hyperbolic orbifolds of dimension 4. We note here that although we will exhibit a family of arithmetic groups with finite index subgroups which embed in a RACG, these groups are not LERF. In [Sun16] Sun has shown that the fundamental group of a cusped arithmetic hyperbolic manifold of dimension $n \geq 4$ is never LERF.

5.1 Rational arithmetic subgroups of $\mathrm{Isom}(\mathbb{H}^4)$

Let q be a rational quadratic form of signature $(4,1)$. It is well known that every indefinite rational quadratic form of rank at least 5 is isotropic (see e.g. [Cas08]). Therefore $\mathbb{H}^4/\mathrm{SO}^+(q,\mathbb{Z})$ is a cusped hyperbolic 4-manifold.

Claim 5.1. *Every rational quadratic form of signature* $(4,1)$ *is equivalent over* $\mathbb{Q}$ *to a quadratic form*

$$Q_{a,b} = -x_0^2 + x_1^2 + x_2^2 + ax_3^2 + bx_4^2$$

where a *and* b *may be taken to be square free integers.*

Proof. There is some vector v with $q(v) = 0$. Pick a positive rational vector w which is orthogonal to v with respect to q and scale it such that $q(w)$ is a square-free integer. Now, $q|_{w^\perp}$, the restriction of q to $w^\perp$, is an isotropic rational quadratic form of signature $(3, 1)$. As such, $q|_{w^\perp}$ is equivalent over $\mathbb{Q}$ to the form $-x_0^2 + x_1^2 + x_2^2 + ax_3^2$ for some a which can be taken to square-free and in $\mathbb{Z}$. Let $b := q(w)$. $\square$

5.2 A family of rational arithmetic groups

Choose two distinct positive primes a, b such that $a, b \not\equiv 3 \mod 4$. A classical result in number theory states that a positive integer can be written as a sum of two squares if and only if its prime decomposition contains no prime congruent to $3 \mod 4$ raised to an odd power. Let

$$a = w^2 + x^2 \text{ and } b = x^2 + z^2. \tag{5.1}$$

Let $Q'_{a,b}$ be the quadratic form

$$Q'_{a,b} := -x_0^2 + x_1^2 + x_2^2 + ax_3^2 + bx_4^2. \tag{5.2}$$

Let $P'_{a,b}$ be the quadratic form $-x_0^2 + x_1^2 + x_2^2 + ax_3^2 + bx_4^2 + b(1+ab)x_5^2 + a(1+ab)x_6^2$. Then $P'_m = Q'_{a,b} \oplus \langle b(1+ab), a(1+ab)\rangle$. The group $\mathrm{SO}^+(Q'_{a,b}, \mathbb{Z})$ is naturally a subgroup of $\mathrm{SO}^+(P'_{a,b}, \mathbb{Z})$.

Lemma 5.2. *The reduction modulo a of the 4th column of an element in* $\mathrm{SO}^+(Q'_{a,b}, \mathbb{Z})$ *is* $(0, 0, 0, \pm 1, 0)$. *The reduction modulo b of the 5th column of an element in* $\mathrm{SO}^+(Q'_{a,b}, \mathbb{Z})$ *is* $(0, 0, 0, 0, \pm 1)$.

Proof. We will prove the second claim. The first claim will follow after conjugation by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Suppose $N' \in \mathrm{SO}^+(Q'_m, \mathbb{Z})$ and $N = \pi_m(N')$. Then N can be written as a block matrix

$$\begin{pmatrix} N_0 & N_1 \\ N_2 & N_3 \end{pmatrix}$$

where N_0 is a 4×4 matrix, N_1 is 4×1, N_2 is 1×4 and N_3 is 1×1. Also, $S_{Q'_{a,b}}$ can be written as a block matrix

$$\begin{pmatrix} S & 0_{4\times 1} \\ 0_{1\times 4} & 0_{1\times 1} \end{pmatrix}$$

where S is a non-singular in $M_4(\mathbb{Z}/b\mathbb{Z})$. Since N must satisfy $N^t \pi_b(S_{Q'_{a,b}})N = \pi_b(S_{Q'_{a,b}})$ we have

$$N^t \pi_m(S_{Q'_{a,b}})N = \begin{pmatrix} N_0^t & N_2^t \\ N_1^t & N_3^t \end{pmatrix} \begin{pmatrix} S & 0_{4\times 1} \\ 0_{1\times 4} & 0_{1\times 1} \end{pmatrix} \begin{pmatrix} N_0 & N_1 \\ N_2 & N_3 \end{pmatrix} \tag{5.3}$$

$$= \begin{pmatrix} N_0^t S N_0 & N_0^t S N_1 \\ N_1^t S N_0 & N_1^t S N_1 \end{pmatrix} \tag{5.4}$$

$$\equiv \begin{pmatrix} S & 0_{4\times 1} \\ 0_{1\times 4} & 0_{1\times 1} \end{pmatrix}. \tag{5.5}$$

Since b is prime, $\mathbb{Z}/b\mathbb{Z}$ is a field and $N_0^t S N_0 \equiv S$ together with $N_0^t S N_1 \equiv 0_{4\times 1}$ implies $N_1 \equiv 0_{1\times 4}$. Therefore, the the 5th column of N' has form $(by_0, by_1, by_2, by_3, y_4)^t$ and must satisfy $-(by_0)^2 + (by_1)^2 + (by_2)^2 + (by_3)^2 + y_4^2 = b$. This implies $y_4^2 = 1 \mod b$ and so $y_4 =\equiv \pm 1 \mod b$ and the 5th column of N' has form $(by_0, by_1, by_2, by_3, \pm 1)^t$. $\square$

Let $\Delta^{a,b}$ be the index 4 subgroup of $\mathrm{SO}^+(Q'_{a,b}, \mathbb{Z})$ of elements whose $(4,4)$-matrix entry is equivalent to 1 modulo a and $(5,5)$-matrix entry is equivalent to 1 modulo b.

Proposition 5.3. *Let a and b be distinct primes with $a, b \not\equiv 3 \mod 4$. The group $\Delta^{a,b}_{(2)}$ embeds in the RACG $\mathrm{SO}^+(F_6, \mathbb{Z})_{(2)}$.*

Proof. Write $a = w^2 + x^2$ and $b = y^2 + z^2$. Consider the 7×7 matrix

$$A'_{a,b} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & bx & -x \\ 0 & 0 & 0 & 0 & y & -z & -az \\ 0 & 0 & 0 & x & 0 & -bw & w \\ 0 & 0 & 0 & 0 & z & y & ay \end{pmatrix} \tag{5.6}$$

with inverse

$$(A'_{a,b})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{w}{a} & 0 & \frac{x}{a} & 0 \\ 0 & 0 & 0 & 0 & \frac{y}{b} & 0 & \frac{z}{b} \\ 0 & 0 & 0 & \frac{x}{1+ab} & -\frac{z}{b(1+ab)} & -\frac{w}{1+ab} & \frac{y}{b(1+ab)} \\ 0 & 0 & 0 & -\frac{x}{a(1+ab)} & -\frac{z}{1+ab} & \frac{w}{a(1+ab)} & \frac{y}{1+ab} \end{pmatrix}. \tag{5.7}$$

Let S_F be the diagonal matrix associated to F_6 and $S_{P'_{a,b}}$ the symmetric matrix associated to $P'_{a,b}$:

$$S_F = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \tag{5.8}$$

$$S_{P'_{a,b}} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(1+ab) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a(1+ab) \end{pmatrix}. \tag{5.9}$$

Then $(A'_{a,b})^t S_F A'_{a,b} = S_{P'_{a,b}}$. Since $A'_{a,b}$ has non-zero determinant $-ab(1+ab)$, it is in $\mathrm{GL}(7,\mathbb{Q})$. Therefore, the forms F_6 and $P'_{a,b}$ are equivalent over $\mathbb{Q}$ and thus

$$A'_{a,b} SO^+(P'_{a,b},\mathbb{Q})(A'_{a,b})^{-1} = SO^+(F_6,\mathbb{Q})$$

with $A'_{a,b} SO^+(P'_{a,b},\mathbb{Q})(A'_{a,b})^{-1}$ and $SO^+(F_6,\mathbb{Z})$ commensurable.

By Lemma 5.2, a matrix N in $\Delta^{a,b}_{(2)}$ sits naturally in $SO^+(P'_{a,b},\mathbb{Z})$ with form

$$\begin{pmatrix} 2a_1+1 & 2b_1 & 2c_1 & 2ad_1 & 2be_1 & 0 & 0 \\ 2a_2 & 2b_2+1 & 2c_2 & 2ad_2 & 2be_2 & 0 & 0 \\ 2a_3 & 2b_3 & 2c_3+1 & 2ad_3 & 2be_3 & 0 & 0 \\ 2a_4 & 2b_4 & 2c_4 & 2ad_4+1 & 2be_4 & 0 & 0 \\ 2a_5 & 2b_5 & 2c_5 & 2ad_5 & 2be_5+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A'_{a,b}N(A'_{a,b})^{-1} = \begin{pmatrix} 1+2a_1 & 2b_1 & 2c_1 & 2d_1w & 2e_1y & 2d_1x & 2e_1z \\ 2a_2 & 1+2b_2 & 2c_2 & 2d_2w & 2e_2y & 2d_2x & 2e_2z \\ 2a_3 & 2b_3 & 1+2c_3 & 2d_3w & 2e_3y & 2d_3x & 2e_3z \\ 2a_4w & 2b_4w & 2c_4w & 1+2ad_4-2d_4x^2 & 2e_4wy & 2d_4wx & 2e_4wz \\ 2a_5y & 2b_5y & 2c_5y & 2d_5wy & 1+2be_5-2e_5z^2 & 2d_5xy & 2e_5yz \\ 2a_4x & 2b_4x & 2c_4x & 2d_4wx & 2e_4xy & 1+2d_4x^2 & 2e_4xz \\ 2a_5z & 2b_5z & 2c_5z & 2d_5wz & 2e_5yz & 2d_5xz & 1+2e_5z^2 \end{pmatrix} \tag{5.10}$$

is in $\mathrm{SO}^+(F_6,\mathbb{Z})_{(2)}$ (see Appendix A.4). $\square$

Appendices

Appendix A

Mathematica computations

This appendix contains computations done by the author using the program Wolfram Mathematica [WR].

A.1 Formulas for φ_m

In this section we compute the formulas for φ_m in Equation 3.4 and Equation 3.5. These computations follow from [JM96, §2].

First we define i, j such that the quaternion algebra $M_2(\mathbb{Q}\sqrt{-m})$ has standard basis $\{1, i, j, ij\}$ and define the elements r, s, t, u.

```
i = {{0, 1}, {-1, 0}}; j = {{1, 0}, {0, -1}};
r = m IdentityMatrix[2]; s = -Sqrt[-m] j;
t = Sqrt[-m]/2 (-i + i.j); u = Sqrt[-m]/2 (i + i.j);
```

The involution τ defined as

$$\tau(a_0 + a_1 i + a_2 j + a_3 ij) = \overline{a_0} - \overline{a_1} i - \overline{a_2} j - \overline{a_3} ij$$

fixes $\mathbb{Q}[r,s,t,u]$ and thus since

$$A = \begin{pmatrix} a_0 + a_1\sqrt{-m} & b_0 + b_1\sqrt{-m} \\ c_0 + c_1\sqrt{-m} & d_0 + d_1\sqrt{-m} \end{pmatrix}$$
$$= \frac{c_0\sqrt{-m} - c_1 m}{m}u + \frac{b_0\sqrt{-m} - b_1}{m}t + \frac{a_0\sqrt{-m} - d_0\sqrt{-m} - a_1 m + d_1 m}{2m}s + \frac{a_0 + d_0 + a_1\sqrt{-m} + d_1\sqrt{-m}}{2m}r$$

then

$$\tau\begin{pmatrix} a_0 + a_1\sqrt{-m} & b_0 + b_1\sqrt{-m} \\ c_0 + c_1\sqrt{-m} & d_0 + d_1\sqrt{-m} \end{pmatrix} = \begin{pmatrix} d_0 - d_1\sqrt{-m} & -b_0 + b_1\sqrt{-m} \\ -c_0 + c_1\sqrt{-m} & a_0 - a_1\sqrt{-m} \end{pmatrix}.$$

```
A={{a0 + a1 Sqrt[-m], b0 + b1 Sqrt[-m]}, {c0 + c1 Sqrt[-m],
  d0 + d1 Sqrt[-m]}};
TA={{d0 - d1 Sqrt[-m], -b0 + b1 Sqrt[-m]}, {-c0 + c1 Sqrt[-m],
  a0 - a1 Sqrt[-m]}};
```

Define $\psi_A(v) = Av\tau(A)$. Then $\varphi_m(A) = \psi_A$.

A.1.1 $m \equiv 1,2 \mod 4$

For $m \equiv 1,2 \mod 4$ we consider the basis $f0, f1, f2, f3$ with $f0 \leftrightarrow u$, $f1 \leftrightarrow t$, $f2 \leftrightarrow s$, $f3 \leftrightarrow u$. Then the ith column of φ_m is determined by $Af_{i+1}\tau(A)$.

```
Solve[A.f0.TA == X0 f0 + X1 f1 + X2 f2 + X3 f3, {X0, X1, X2, X3}]
{{X0 -> d0^2 + d1^2 m, X1 -> -b0^2 - b1^2 m, X2 -> b0 d0 + b1 d1 m,
  X3 -> b1 d0 - b0 d1}}

Solve[A.f1.TA == X0 f0 + X1 f1 + X2 f2 + X3 f3, {X0, X1, X2, X3}]
{{X0 -> -c0^2 - c1^2 m, X1 -> a0^2 + a1^2 m, X2 -> -a0 c0 - a1 c1 m,
  X3 -> -a1 c0 + a0 c1}}
```

```
Solve[A.f2.TA == X0 f0 + X1 f1 + X2 f2 + X3 f3, {X0, X1, X2, X3}]
{{X0 -> 2 (c0 d0 + c1 d1 m), X1 -> -2 (a0 b0 + a1 b1 m),
  X2 -> b0 c0 + a0 d0 + b1 c1 m + a1 d1 m,
  X3 -> b1 c0 - b0 c1 + a1 d0 - a0 d1}}

Solve[A.f3.TA == X0 f0 + X1 f1 + X2 f2 + X3 f3, {X0, X1, X2, X3}]
{{X0 -> -2 (c1 d0 m - c0 d1 m), X1 -> 2 (a1 b0 m - a0 b1 m),
  X2 -> b1 c0 m - b0 c1 m - a1 d0 m + a0 d1 m,
  X3 -> -b0 c0 + a0 d0 - b1 c1 m + a1 d1 m}}
```

Putting these together, we get Equation 3.4.

A.1.2 $m \equiv 3 \mod 4$

Set

$$B = \begin{pmatrix} a_0 + a_1\frac{1+\sqrt{-m}}{2} & b_0 + b_1\frac{1+\sqrt{-m}}{2} \\ c_0 + c_1\frac{1+\sqrt{-m}}{2} & d_0 + d_1\frac{1+\sqrt{-m}}{2} \end{pmatrix}$$

and

$$TB = \tau(B) = \begin{pmatrix} d_0 + d_1\frac{1-\sqrt{-m}}{2} & -b_0 - b_1\frac{1-\sqrt{-m}}{2} \\ -c_0 - c_1\frac{1-\sqrt{-m}}{2} & d_0 + d_1\frac{1-\sqrt{-m}}{2} \end{pmatrix}$$

For $m \equiv 3 \mod 4$ we consider the basis $f0, f1, f2, f3$ with $f0 \leftrightarrow u$, $f1 \leftrightarrow t$, $f2 \leftrightarrow s$, $f3 \leftrightarrow -\frac{r-s}{2}$. Then the ith column of φ_m is determined by $Bf_{i+1}\tau(B)$.

```
Solve[B.f0.TB == X0 f0 + X1 f1 + X2 f2 + X3 f3, {X0, X1, X2, X3}]
{{X0 -> 1/4 (4 d0^2 + 4 d0 d1 + d1^2 + d1^2 m),
  X1 -> 1/4 (-4 b0^2 - 4 b0 b1 - b1^2 - b1^2 m),
  X2 -> 1/4 (4 b0 d0 + 4 b1 d0 + b1 d1 + b1 d1 m),
  X3 -> -b1 d0 + b0 d1}}

Solve[B.f1.TB == X0 f0 + X1 f1 + X2 f2 + X3 f3, {X0, X1, X2, X3}]
{{X0 -> 1/4 (-4 c0^2 - 4 c0 c1 - c1^2 - c1^2 m),
  X1 -> 1/4 (4 a0^2 + 4 a0 a1 + a1^2 + a1^2 m),
  X2 -> 1/4 (-4 a0 c0 - 4 a1 c0 - a1 c1 - a1 c1 m),
  X3 -> a1 c0 - a0 c1}}

Solve[B.f2.TB == X0 f0 + X1 f1 + X2 f2 + X3 f3, {X0, X1, X2, X3}]
```

```
{{X0 -> 1/2 (4 c0 d0 + 2 c1 d0 + 2 c0 d1 + c1 d1 + c1 d1 m),
  X1 -> 1/2 (-4 a0 b0 - 2 a1 b0 - 2 a0 b1 - a1 b1 - a1 b1 m),
  X2 -> 1/4 (4 b0 c0 + 4 b1 c0 + b1 c1 + 4 a0 d0 + 4 a1 d0 + a1 d1 +
      b1 c1 m + a1 d1 m), X3 -> -b1 c0 + b0 c1 - a1 d0 + a0 d1}}

Solve[B.f3.TB == X0 f0 + X1 f1 + X2 f2 + X3 f3, {X0, X1, X2, X3}]
{{X0 -> 1/
    4 (4 c0 d0 + 2 c1 d0 + 2 c0 d1 + c1 d1 + 2 c1 d0 m - 2 c0 d1 m +
      c1 d1 m),
  X1 -> 1/4 (-4 a0 b0 - 2 a1 b0 - 2 a0 b1 - a1 b1 - 2 a1 b0 m +
      2 a0 b1 m - a1 b1 m),
  X2 -> 1/4 (4 b0 c0 + 3 b1 c0 + b0 c1 + b1 c1 + a1 d0 - a0 d1 -
      b1 c0 m + b0 c1 m + b1 c1 m + a1 d0 m - a0 d1 m),
  X3 -> 1/4 (-4 b0 c0 - 4 b1 c0 - b1 c1 + 4 a0 d0 + 4 a0 d1 + a1 d1 -
      b1 c1 m + a1 d1 m)}}
```

Putting these together , replacing m with $4k-1$, and simplifying with the assumption that the determinant of B is 1, we get Equation 3.5.

Recall that $det(B) = 1$.

```
Collect[Det[B], Sqrt[-m]]
-b0 c0 - (b1 c0)/2 - (b0 c1)/2 - (b1 c1)/4 + a0 d0 +(a1 d0)/2 + (a0 d1)/2 + (
     a1 d1)/4 + (-((b1 c0)/2) - (b0 c1)/2 - (b1 c1)/2 + (a1 d0)/2 + (a0 d1)/2
    + (a1 d1)/2) Sqrt[-m] + (b1 c1 m)/4 - (a1 d1 m)/4
```

In particular, note $-b1c0 - b0c1 - b1c1 + a1d0 + a0d1 + a1d1 = 0$.

A.1.3 Comment on formulas for φ_m

Alternative to the computation of φ_m as in [JM96], one might also compute as in [EGM98, §1.3]. We comment here that following the computations herein leads to a formula which differs from the one found in [EGM98, Proposition 1.3.11], i.e. the formula in [EGM98, Proposition 1.3.11] was computed incorrectly. This incorrect formula is then used in [EGM98, p. 463] to compute a formula for a map $\Psi : \mathrm{PSL}(2, \mathbb{Q}\sqrt{-m}) \to \mathrm{O}(q, \mathbb{Q})$.

It is easy to see that the Ψ in [EGM98, p. 463] is incorrect by the evaluation $\Psi(\alpha)$ where

$$\alpha = \begin{pmatrix} 0 & w \\ -\frac{1}{w} & 0 \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Q}\sqrt{-m}),\ w = 1 + \sqrt{-m}.$$

In particular, $det(\alpha) = 1$ but $det(\Psi(\alpha)) = (1 - 6m + m^2)/(1 + m)^2 \neq 1$.

Below we demonstrate how to follow the computations in [EGM98, §1.3] and arrive at the correct map ψ which will differ from the incorrectly computed formula [EGM98, Proposition 1.3.11]. These computations were made using the NCAlgebra package for Mathematica.

To use NCAlgebra package use this command to install:

```
Import["https://raw.githubusercontent.com/NCAlgebra/NC/master/NCExtras/
    NCWebInstall.m"];
```

Use the command below to run:

```
<< NC`
```

Use the command below to load:

```
<< NCAlgebra`

SetCommutative[a0, a1, b0, b1, c0, c1, d0, d1, m];
SNC[f0, f1, f2, f3];
```

We have the following rules for f0, f1, f2, f3:

```
L = {f3 ** f2 -> -f2 ** f3, f3 ** f1 -> -f1 ** f3, f3 ** f0 -> -f0 ** f3, f2
    ** f1 -> -f1 ** f2, f2 ** f0 -> -f0 ** f2, f1 ** f0 -> -f0 ** f1, f0 ** f0
     -> 1, f1 ** f1 -> -1, f2 ** f2 -> -1, f3 ** f3 -> -m};
```

Here we define A:= A and AS:= A^* as in [EGM98, p. 16]

```
A = 1/2 (a0 + d0 + (a0 - d0) f0 ** f1 + (b0 + c0) f0 ** f2 + (b1 - c1) f0 **
    f3 + (c0 - b0) f1 ** f2 - (b1 + c1) f1 ** f3 + (a1 - d1) f2 ** f3 + (a1 +
    d1) f0 ** f1 ** f2 ** f3);
AS = 1/2 (a0 + d0 - (a0 - d0) f0 ** f1 - (b0 + c0) f0 ** f2 - (b1 - c1) f0 **
     f3 - (c0 - b0) f1 ** f2 + (b1 + c1) f1 ** f3 - (a1 - d1) f2 ** f3 + (a1 +
     d1) f0 ** f1 ** f2 ** f3);
```

The rows of Ψ are then determined as follows (output ommited):

```
NCReplaceRepeated[NCExpand[A ** f0 ** AS], L]
NCReplaceRepeated[NCExpand[A ** f1 ** AS], L]
NCReplaceRepeated[NCExpand[A ** f2 ** AS], L]
NCReplaceRepeated[NCExpand[A ** f3 ** AS], L]
```

Putting them together we get the following correct formula:

```
Psi = {{a0^2/2 + b0^2/2 + c0^2/2 + d0^2/2 + (a1^2 m)/2 + (b1^2 m)/2 + (c1^2 m
    )/2 + (d1^2 m)/2, -(a0^2/2) - b0^2/2 + c0^2/2 + d0^2/2 - (a1^2 m)/2 - (b1
    ^2 m)/2 + (c1^2 m)/2 + (d1^2 m)/2, -a0 c0 - b0 d0 - a1 c1 m - b1 d1 m, -a1
     c0 + a0 c1 - b1 d0 + b0 d1}, {-(a0^2/2) + b0^2/2 - c0^2/2 + d0^2/2 - (a1
    ^2 m)/2 + (b1^2 m)/2 - (c1^2 m)/2 + (d1^2 m)/2, a0^2/2 - b0^2/2 - c0^2/2 +
     d0^2/2 + (a1^2 m)/2 - (b1^2 m)/2 - (c1^2 m)/2 + (d1^2 m)/2, a0 c0 - b0 d0
     + a1 c1 m - b1 d1 m, a1 c0 - a0 c1 - b1 d0 + b0 d1}, {-a0 b0 - c0 d0 - a1
     b1 m - c1 d1 m, a0 b0 - c0 d0 + a1 b1 m - c1 d1 m, b0 c0 + a0 d0 + b1 c1
    m + a1 d1 m, b1 c0 - b0 c1 + a1 d0 - a0 d1}, {a1 b0 m - a0 b1 m + c1 d0 m
    - c0 d1 m, -a1 b0 m + a0 b1 m + c1 d0 m - c0 d1 m, b1 c0 m - b0 c1 m - a1
    d0 m + a0 d1 m, -b0 c0 + a0 d0 - b1 c1 m + a1 d1 m}}
```

This formula corresponds to the quadratic form $x_0^2 - x_1^2 - x_2^2 - mx_3^2$. Therefore, we must conjugate it by C12 or C3 to preserve the forms we want.

```
C12 = {{1/2, -(1/2), 0, 0}, {-(1/2), -(1/2), 0, 0}, {0, 0, 1, 0}, {0, 0, 0,
    1}};
C3 = {{1/2, -(1/2), 0, 0}, {-(1/2), -(1/2), 0, 0}, {0, 0, 1, 0}, {0, 0, 1/2,
    1/2}};
```

The formulas `C12.Psi.Inverse[C12]` and `C3.Psi.Inverse[C3]` give the same formulas for Equation 3.4 and Equation 3.5 as above.

A.2 Computations in the proof of Theorem 3.1

A.2.1 $m \equiv 1, 2 \mod 4$

A matrix N in $\mathrm{PSL}(2, \mathcal{O}_m)_{(2)}$ has form

$$\begin{pmatrix} 2a_0 + 1 + 2a_1\frac{1+\sqrt{-m}}{2} & 2b_0 + 2b_1\frac{1+\sqrt{-m}}{2} \\ 2c_0 + 2c_1\frac{1+\sqrt{-m}}{2} & 2d_0 + 1 + 2d_1\frac{1+\sqrt{-m}}{2} \end{pmatrix},$$

and has determinant 1. We can compute the determinant

```
Collect[Det[{{1 + 2 a0 + 2 a1 Sqrt[-m], 2 b0 + 2 b1 Sqrt[-m], {2 c0 + 2 c1
    Sqrt[-m], 1 + 2 d0 + 2 d1 Sqrt[-m]}}], Sqrt[-m]]
1 + 2 a0 - 4 b0 c0 + 2 d0 + 4 a0 d0 + (2 a1 - 4 b1 c0 - 4 b0 c1 + 4 a1 d0 + 2
     d1 + 4 a0 d1) Sqrt[-m] + 4 b1 c1 m - 4 a1 d1 m
```

We can separate the real part from the imaginary part and set $m = 4k - 1$.

```
D1 =
D0 =
```

Note that we must have D1=1 and D0=0.

```
Factor[a0 - 2 b0 c0 + d0 + 2 a0 d0 - (D1 - 1)/2]
-2 (b1 c1 - a1 d1) m
```

So we know $a0 - 2b0c0 + d0 + 2a0d0 = -2(b1c1 - a1d1)m$. We set two replacement rules for later use.

```
L = a0 - 2 b0 c0 + d0 + 2 a0 d0 -> -2 (b1 c1 - a1 d1) m;
L1 = -8 b0 c0 + 4 d0 + a0 (4 + 8 d0) -> -8 (b1 c1 - a1 d1) m;
```

We set A12 and A12I to be the matrices in Equation 3.9 and Equation 3.10.

```
A12={{1/2, -(1/2), 0, 0, 0, 0, 0}, {1/2, 1/2, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0,
     0, 0}, {0, 0, 0, w, -x, -y, -z}, {0, 0, 0, x, w, z, -y}, {0, 0, 0, y, -z,
     w, x}, {0, 0, 0, z, y, -x, w}};
A12I={{1, 1, 0, 0, 0, 0, 0}, {-1, 1, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0},
    {0, 0, 0, w/m, x/m, y/m, z/m}, {0, 0, 0, -(x/m), w/m, -(z/m), y/m}, {0, 0,
     0, -(y/m), z/m, w/m, -(x/m)}, {0, 0, 0, -(z/m), -(y/m), x/m, w/m}};
```

Next we compute $\varphi_m(N)$.

```
phi12[{{a0_, a1_}, {b0_, b1_}, {c0_, c1_}, {d0_, d1_}}]
  = {{d0^2 + d1^2 m, -c0^2 - c1^2 m, 2 (c0 d0 + c1 d1 m), -2 (c1 d0 m - c0 d1
      m)}, {-b0^2 - b1^2 m, a0^2 + a1^2 m, -2 (a0 b0 + a1 b1 m), 2 (a1 b0 m -
      a0 b1 m)}, {b0 d0 + b1 d1 m, -a0 c0 - a1 c1 m, b0 c0 + a0 d0 + b1 c1 m
     + a1 d1 m, b1 c0 m - b0 c1 m - a1 d0 m + a0 d1 m}, {b1 d0 - b0 d1, -a1
     c0 + a0 c1, b1 c0 - b0 c1 + a1 d0 - a0 d1, -b0 c0 + a0 d0 - b1 c1 m + a1
      d1 m}};

Nmat = phi12[{{1 + 2 a0, 2 a1}, {2 b0, 2 b1}, {2 c0, 2 c1}, {1 + 2 d0, 2 d1
    }}];
```

We now extend $\varphi_m(N)$, conjugate, and simplify the expression in several steps. Note that attempting to simplify all at once did not seem to work for me.

```
Simp1 = FullSimplify[A12.ArrayFlatten[{{Nmat, ConstantArray[0, {4, 3}]}, {
    ConstantArray[0, {3, 4}], IdentityMatrix[3]}}].A12I , D1 == 1 && D0 == 0
    && w^2 + x^2 + y^2 + z^2 == m];

Sim2 = Simplify[Sim1 /. L];

Sim3 = Simplify[Sim2 /. L1];

Sim4 = Simplify[Factor[Sim3] /. -w^2 - x^2 - y^2 - z^2 -> -m]
{{1 + 2 a0 + 2 a0^2 + 2 b0^2 + 2 c0^2 + 2 d0 + 2 d0^2 + 2 a1^2 m + 2 b1^2 m +
     2 c1^2 m + 2 d1^2 m, -2 (a0 + a0^2 - b0^2 + c0^2 - d0 - d0^2 + a1^2 m -
    b1^2 m + c1^2 m - d1^2 m), 2 (b0 + 2 a0 b0 + c0 + 2 c0 d0 + 2 (a1 b1 + c1
    d1) m), -2 (2 a1 b0 - (1 + 2 a0) b1 + c1 + 2 c1 d0 - 2 c0 d1) w, -2 (2 a1
    b0 - (1 + 2 a0) b1 + c1 + 2 c1 d0 - 2 c0 d1) x, -2 (2 a1 b0 - (1 + 2 a0)
    b1 + c1 + 2 c1 d0 - 2 c0 d1) y, -2 (2 a1 b0 - (1 + 2 a0) b1 + c1 + 2 c1 d0
     - 2 c0 d1) z}, {-2 (a0 + a0^2 + b0^2 - c0^2 - d0 - d0^2 + a1^2 m + b1^2 m
     - c1^2 m - d1^2 m), 1 + 2 a0 + 2 a0^2 - 2 b0^2 - 2 c0^2 + 2 d0 + 2 d0^2 +
     2 a1^2 m - 2 b1^2 m - 2 c1^2 m + 2 d1^2 m, -2 b0 - 4 a0 b0 + 2 c0 + 4 c0
    d0 - 4 a1 b1 m + 4 c1 d1 m, 2 (2 a1 b0 - (1 + 2 a0) b1 - c1 - 2 c1 d0 + 2
    c0 d1) w, 2 (2 a1 b0 - (1 + 2 a0) b1 - c1 - 2 c1 d0 + 2 c0 d1) x, 2 (2 a1
    b0 - (1 + 2 a0) b1 - c1 - 2 c1 d0 + 2 c0 d1) y, 2 (2 a1 b0 - (1 + 2 a0) b1
     - c1 - 2 c1 d0 + 2 c0 d1) z}, {2 (b0 + c0 + 2 a0 c0 + 2 b0 d0 + 2 (a1 c1
    + b1 d1) m), 2 (b0 - (1 + 2 a0) c0 + 2 b0 d0 - 2 a1 c1 m + 2 b1 d1 m), 1 +
     8 b0 c0 + 8 a1 d1 m, 4 (-2 b0 c1 + d1 + 2 a0 d1) w, 4 (-2 b0 c1 + d1 + 2
    a0 d1) x, 4 (-2 b0 c1 + d1 + 2 a0 d1) y, 4 (-2 b0 c1 + d1 + 2 a0 d1) z},
    {2 (b1 + 2 a1 c0 - c1 - 2 a0 c1 + 2 b1 d0 - 2 b0 d1) w, 2 (b1 - 2 a1 c0 +
```

```
c1 + 2 a0 c1 + 2 b1 d0 - 2 b0 d1) w, 4 (a1 - 2 b0 c1 + 2 a1 d0) w, 1 - 8
b1 c1 w^2 + 8 a1 d1 w^2, -8 (b1 c1 - a1 d1) w x, -8 (b1 c1 - a1 d1) w y,
-8 (b1 c1 - a1 d1) w z}, {2 (b1 + 2 a1 c0 - c1 - 2 a0 c1 + 2 b1 d0 - 2 b0
d1) x, 2 (b1 - 2 a1 c0 + c1 + 2 a0 c1 + 2 b1 d0 - 2 b0 d1) x, 4 (a1 - 2 b0
 c1 + 2 a1 d0) x, -8 (b1 c1 - a1 d1) w x, 1 - 8 b1 c1 x^2 + 8 a1 d1 x^2,
-8 (b1 c1 - a1 d1) x y, -8 (b1 c1 - a1 d1) x z}, {2 (b1 + 2 a1 c0 - c1 - 2
 a0 c1 + 2 b1 d0 - 2 b0 d1) y, 2 (b1 - 2 a1 c0 + c1 + 2 a0 c1 + 2 b1 d0 -
2 b0 d1) y, 4 (a1 - 2 b0 c1 + 2 a1 d0) y, -8 (b1 c1 - a1 d1) w y, -8 (b1
c1 - a1 d1) x y, 1 - 8 b1 c1 y^2 + 8 a1 d1 y^2, -8 (b1 c1 - a1 d1) y z},
{2 (b1 + 2 a1 c0 - c1 - 2 a0 c1 + 2 b1 d0 - 2 b0 d1) z, 2 (b1 - 2 a1 c0 +
c1 + 2 a0 c1 + 2 b1 d0 - 2 b0 d1) z, 4 (a1 - 2 b0 c1 + 2 a1 d0) z, -8 (b1
c1 - a1 d1) w z, -8 (b1 c1 - a1 d1) x z, -8 (b1 c1 - a1 d1) y z, 1 - 8 b1
c1 z^2 + 8 a1 d1 z^2}}
```

This shows $A_m N' A_m^{-1} \in \mathrm{SO}^+(F_6, \mathbb{Z})$.

We now check that the reduction modulo 2 is the identity.

```
PolynomialMod[Sim4, 2] == IdentityMatrix[7]
True
```

A.2.2 $m \equiv 3 \mod 4$

A matrix N in $\mathrm{PSL}(2, \mathcal{O}_m)_{(2)}$ has form

$$\begin{pmatrix} 2a_0 + 1 + 2a_1 \frac{1+\sqrt{-m}}{2} & 2b_0 + 2b_1 \frac{1+\sqrt{-m}}{2} \\ 2c_0 + 2c_1 \frac{1+\sqrt{-m}}{2} & 2d_0 + 1 + 2d_1 \frac{1+\sqrt{-m}}{2} \end{pmatrix},$$

and has determinant 1. We can compute the determinant

```
Collect[Det[{{1 + 2 a0 + 2 a1 (1 + Sqrt[-m])/2, 2 b0 + 2 b1 (1 + Sqrt[-m])
   /2}, {2 c0 + 2 c1 (1 + Sqrt[-m])/2, 1 + 2 d0 + 2 d1 (1 + Sqrt[-m])/2}}],
   Sqrt[-m]]
1 + 2 a0 + a1 - 4 b0 c0 - 2 b1 c0 - 2 b0 c1 - b1 c1 + 2 d0 + 4 a0 d0 + 2 a1
   d0 + d1 + 2 a0 d1 + a1 d1 + (a1 - 2 b1 c0 - 2 b0 c1 - 2 b1 c1 + 2 a1 d0 +
   d1 + 2 a0 d1 + 2 a1 d1) Sqrt[-m] + b1 c1 m - a1 d1 m
```

We can separate the real part from the imaginary part and set $m = 4k - 1$.

```
D1 = 1 + 2 a0 + a1 - 4 b0 c0 - 2 b1 c0 - 2 b0 c1 - b1 c1 + 2 d0 + 4 a0 d0 + 2
    a1 d0 + d1 + 2 a0 d1 + a1 d1 + b1 c1 m - a1 d1 m /.m -> 4 k - 1;
D0 = (a1 - 2 b1 c0 - 2 b0 c1 - 2 b1 c1 + 2 a1 d0 + d1 + 2 a0 d1 + 2 a1 d1) /.
    m -> 4 k - 1;
```

Note that we must have D1=1 and D0=0.

```
Factor[DC0 + 1 - DC1 + (-4 b0 c0 + b1 c1 + 2 d0 + a0 (2 + 4 d0) - a1 d1)]
-(b1 c1 - a1 d1) (-1 + 4 k)
```

So we know $(-4b0c0+b1c1+2d0+a0(2+4d0)-a1d1) = -(b1c1-a1d1)(-1+4k)$. We set two replacement rules for later use.

```
L = (-4 b0 c0 + b1 c1 + 2 d0 + a0 (2 + 4 d0) - a1 d1) -> -(b1 c1 - a1 d1) (-1
     + 4 k);
L1 = -8 b0 c0 + 2 b1 c1 + 4 d0 + a0 (4 + 8 d0) - 2 a1 d1 -> -2 (b1 c1 - a1 d1
    ) (-1 + 4 k);
```

We set A3 and A3I to be the matrices in Equation 3.15 and Equation 3.16.

```
A3={{1/2, -(1/2), 0, 0, 0, 0, 0}, {1/2, 1/2, 0, 0, 0, 0, 0}, {0, 0, 1, 1/2,
    0, 0, 0}, {0, 0, 0, -(w/2), -x, -y, -z}, {0, 0, 0, -(x/2), w, z, -y}, {0,
    0, 0, -(y/2), -z, w, x}, {0, 0, 0, -(z/2), y, -x, w}};
A3I={{1, 1, 0, 0, 0, 0, 0}, {-1, 1, 0, 0, 0, 0, 0}, {0, 0, 1, w/m, x/m, y/m,
    z/m}, {0, 0, 0, -((2 w)/m), -((2 x)/m), -((2 y)/m), -((2 z)/m)}, {0, 0, 0,
     -(x/m), w/m, -(z/m), y/m}, {0, 0, 0, -(y/m), z/m, w/m, -(x/m)}, {0, 0, 0,
     -(z/m), -(y/m), x/m, w/m}};
```

Next we compute $\varphi_m(N)$.

```
phi3[{{a0_, a1_}, {b0_, b1_}, {c0_, c1_}, {d0_, d1_}}]
  = {{d0^2 + d0 d1 + d1^2 k, -c0^2 - c0 c1 - c1^2 k, 2 c0 d0 + c1 d0 + c0 d1
      + 2 c1 d1 k, c0 d0 + c0 d1 + 2 c1 d0 k - 2 c0 d1 k + c1 d1 k}, {-b0^2 -
      b0 b1 -  b1^2 k, a0^2 + a0 a1 + a1^2 k, -2 a0 b0 - a1 b0 - a0 b1 - 2 a1
      b1 k, -a0 b0 - a0 b1 - 2 a1 b0 k + 2 a0 b1 k - a1 b1 k}, {b0 d0 + b1 d0
      + b1 d1 k, -a0 c0 - a1 c0 - a1 c1 k, b0 c0 + b1 c0 + a0 d0 + a1 d0 + b1
      c1 k + a1 d1 k, b0 c0 + b1 c0 - b1 c0 k + b0 c1 k + b1 c1 k + a1 d0 k -
      a0 d1 k}, {-b1 d0 + b0 d1, a1 c0 - a0 c1, -b1 c0 + b0 c1 - a1 d0 + a0 d1
      , -b0 c0 - b1 c0 + a0 d0 + a0 d1 - b1 c1 k + a1 d1 k}};

Nmat = phi3[{{1 + 2 a0, 2 a1}, {2 b0, 2 b1}, {2 c0, 2 c1}, {1 + 2 d0, 2 d1
    }}];
```

We now extend $\varphi_m(N)$, conjugate, and simplify the expression in several steps. Note that attempting to simplify all at once did not seem to work for me.

```
Simp1 = FullSimplify[A3.ArrayFlatten[{{Nmat, ConstantArray[0, {4, 3}]}, {
    ConstantArray[0, {3, 4}], IdentityMatrix[3]}}].A3I /. m -> 4 k - 1, D0 ==
    0 && D1 == 1];

Simp2 = FullSimplify[Simp1, w^2 + x^2 + y^2 + z^2 == 4 k - 1];

Simp3 = Simp2 /. L;

Simp4 = Simp3 /. L1;

Simp5 = FullSimplify[Factor[Simp4] /. -w^2 - x^2 - y^2 - z^2 -> -(4 k - 1)]
{{1 + a1 + 2 a0 (1 + a0 + a1) + 2 b0 (b0 + b1) + 2 c0 (c0 + c1) + d1 + 2 d0
    (1 + d0 + d1) + 2 (a1^2 + b1^2 + c1^2 + d1^2) k, -a1 - 2 a0 (1 + a0 + a1)
    + 2 b0 (b0 + b1) - 2 c0 (c0 + c1) + d1 + 2 d0 (1 + d0 + d1) + 2 (-a1^2 +
    b1^2 - c1^2 + d1^2) k, 2 (1 + 2 a0 + a1) b0 + b1 + 2 a0 b1 + c1 + 2 c1 d0
    + 2 c0 (1 + 2 d0 + d1) + 4 a1 b1 k + 4 c1 d1 k, (-2 a1 b0 + b1 + 2 a0 b1 -
     c1 - 2 c1 d0 + 2 c0 d1) w, (-2 a1 b0 + b1 + 2 a0 b1 - c1 - 2 c1 d0 + 2 c0
     d1) x, (-2 a1 b0 + b1 + 2 a0 b1 - c1 - 2 c1 d0 + 2 c0 d1) y, (-2 a1 b0 +
    b1 + 2 a0 b1 - c1 - 2 c1 d0 + 2 c0 d1) z}, {-a1 - 2 a0 (1 + a0 + a1) - 2
    b0 (b0 + b1) + 2 c0 (c0 + c1) + d1 + 2 d0 (1 + d0 + d1) + 2 (-a1^2 - b1^2
    + c1^2 + d1^2) k, 1 + a1 + 2 a0 (1 + a0 + a1) - 2 b0 (b0 + b1) - 2 c0 (c0
    + c1) + d1 + 2 d0 (1 + d0 + d1) + 2 (a1^2 - b1^2 - c1^2 + d1^2) k, -2 (1 +
     2 a0 + a1) b0 + c1 + 2 c1 d0 + 2 c0 (1 + 2 d0 + d1) + 4 c1 d1 k - b1 (1 +
     2 a0 + 4 a1 k), (2 a1 b0 - (1 + 2 a0) b1 - c1 - 2 c1 d0 + 2 c0 d1) w, (2
    a1 b0 - (1 + 2 a0) b1 - c1 - 2 c1 d0 + 2 c0 d1) x, (2 a1 b0 - (1 + 2 a0)
    b1 - c1 - 2 c1 d0 + 2 c0 d1) y, (2 a1 b0 - (1 + 2 a0) b1 - c1 - 2 c1 d0 +
    2 c0 d1) z}, {c1 + 2 b0 (1 + 2 d0 + d1) + 2 (c0 + 2 a0 c0 + a1 c0 + a0 c1
    + 2 a1 c1 k) + b1 (1 + 2 d0 + 4 d1 k), -2 (1 + 2 a0 + a1) c0 + 2 b0 (1 + 2
     d0 + d1) - c1 (1 + 2 a0 + 4 a1 k) + b1 (1 + 2 d0 + 4 d1 k), 1 + 8 b0 c0 -
     2 b1 c1 + 2 d1 + 4 a0 d1 + 2 a1 (1 + 2 d0 + d1 + 4 d1 k), -2 (2 b0 c1 +
    b1 c1 - (1 + 2 a0 + a1) d1) w, -2 (2 b0 c1 + b1 c1 - (1 + 2 a0 + a1) d1) x
    , -2 (2 b0 c1 + b1 c1 - (1 + 2 a0 + a1) d1) y, -2 (2 b0 c1 + b1 c1 - (1 +
    2 a0 + a1) d1) z}, {(b1 + 2 a1 c0 - c1 - 2 a0 c1 + 2 b1 d0 - 2 b0 d1) w, (
    b1 - 2 a1 c0 + c1 + 2 a0 c1 + 2 b1 d0 - 2 b0 d1) w, 2 (-(2 b0 + b1) c1 +
    a1 (1 + 2 d0 + d1)) w, 1 + 2 (-b1 c1 + a1 d1) w^2, 2 (-b1 c1 + a1 d1) w x,
     2 (-b1 c1 + a1 d1) w y, 2 (-b1 c1 + a1 d1) w z}, {(b1 + 2 a1 c0 - c1 - 2
    a0 c1 + 2 b1 d0 - 2 b0 d1) x, (b1 - 2 a1 c0 + c1 + 2 a0 c1 + 2 b1 d0 - 2
    b0 d1) x, 2 (-(2 b0 + b1) c1 + a1 (1 + 2 d0 + d1)) x, 2 (-b1 c1 + a1 d1) w
     x, 1 + 2 (-b1 c1 + a1 d1) x^2, 2 (-b1 c1 + a1 d1) x y, 2 (-b1 c1 + a1 d1)
     x z}, {(b1 + 2 a1 c0 - c1 - 2 a0 c1 + 2 b1 d0 - 2 b0 d1) y, (b1 - 2 a1 c0
     + c1 + 2 a0 c1 + 2 b1 d0 - 2 b0 d1) y, 2 (-(2 b0 + b1) c1 + a1 (1 + 2 d0
    + d1)) y, 2 (-b1 c1 + a1 d1) w y, 2 (-b1 c1 + a1 d1) x y, 1 + 2 (-b1 c1 +
    a1 d1) y^2, 2 (-b1 c1 + a1 d1) y z}, {(b1 + 2 a1 c0 - c1 - 2 a0 c1 + 2 b1
    d0 - 2 b0 d1) z, (b1 - 2 a1 c0 + c1 + 2 a0 c1 + 2 b1 d0 - 2 b0 d1) z, 2
    (-(2 b0 + b1) c1 + a1 (1 + 2 d0 + d1)) z, 2 (-b1 c1 + a1 d1) w z, 2 (-b1
    c1 + a1 d1) x z, 2 (-b1 c1 + a1 d1) y z, 1 + 2 (-b1 c1 + a1 d1) z^2}}
```

This shows $A_m N' A_m^{-1} \in \mathrm{SO}^+(F_6, \mathbb{Z})$.

Before we check the reduction modulo 2 of Simp5 we make a simplification.

```
PolynomialMod[DC0, 2]
a1 + d1
```

So $\mod 2$ we have $a1 \equiv d1$.

```
PolynomialMod[Simp5 /. d1 -> a1, 2]
True
```

```
PolynomialMod[Sim4, 2] == IdentityMatrix[7]
{{1, 0, b1 + c1, b1 w + c1 w, b1 x + c1 x, b1 y + c1 y, b1 z + c1 z}, {0, 1,
    b1 + c1, b1 w + c1 w, b1 x + c1 x, b1 y + c1 y, b1 z + c1 z}, {b1 + c1, b1
     + c1, 1, 0, 0, 0, 0}, {b1 w + c1 w, b1 w + c1 w, 0, 1, 0, 0, 0}, {b1 x +
    c1 x, b1 x + c1 x, 0, 0, 1, 0, 0}, {b1 y + c1 y, b1 y + c1 y, 0, 0, 0, 1,
    0}, {b1 z + c1 z, b1 z + c1 z, 0, 0, 0, 0, 1}}
```

This is indeed Equation 3.20.

A.3 Computations in the proof of Proposition 4.3

We set A and AI to be the matrices in Equation 4.5 and Equation 4.6.

```
A={{4 k, 0, 0, 0, 1 - 4 k}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1,
    0}, {1 - 4 k, 0, 0, 0, 4 k}};
AI={{(4 k)/(-1 + 8 k), 0, 0, 0, (-1 + 4 k)/(-1 + 8 k)}, {0, 1, 0, 0, 0}, {0,
    0, 1, 0, 0}, {0, 0, 0, 1, 0}, {(-1 + 4 k)/(-1 + 8 k), 0, 0, 0, (4 k)/(-1 +
     8 k)}};
```

Next we check $(A'_m)^t S_F A'_m = S_{P'_m}$.

```
Simplify[Transpose[A].DiagonalMatrix[{-1, 1, 1, 1, 1}].A == DiagonalMatrix[{-
    m, 1, 1, 1, m}]]
True
```

A matrix N in $\Delta^m_{(2)}$ has form

$$\begin{pmatrix} 2ma_1+1 & 2b_1 & 2c_1 & 2d_1 & 0 \\ 2ma_2 & 2b_2+1 & 2c_2 & 2d_2 & 0 \\ 2ma_3 & 2b_3 & 2c_3+1 & 2d_3 & 0 \\ 2ma_4 & 2b_4 & 2c_4 & 2d_4+1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We compute $A'_m N (A'_m)^{-1}$.

```
Nmat = {{1 + 2 a1 (-1 + 8 k), 2 b1, 2 c1, 2 d1, 0}, {2 a2 (-1 + 8 k), 1 + 2
    b2, 2 c2, 2 d2, 0}, {2 a3 (-1 + 8 k), 2 b3, 1 + 2 c3, 2 d3, 0}, {2 a4 (-1
    + 8 k), 2 b4, 2 c4, 1 + 2 d4, 0}, {0, 0, 0, 0, 1}};

Expand[Simplify[A.Nmat.AI]]
{{1 + 32 a1 k^2, 8 b1 k, 8 c1 k, 8 d1 k, -8 a1 k + 32 a1 k^2}, {8 a2 k, 1 + 2
     b2, 2 c2, 2 d2, -2 a2 + 8 a2 k}, {8 a3 k, 2 b3, 1 + 2 c3, 2 d3, -2 a3 + 8
     a3 k}, {8 a4 k, 2 b4, 2 c4, 1 + 2 d4, -2 a4 + 8 a4 k}, {8 a1 k - 32 a1 k
    ^2, 2 b1 - 8 b1 k, 2 c1 - 8 c1 k, 2 d1 - 8 d1 k, 1 - 2 a1 + 16 a1 k - 32
    a1 k^2}}
```

A.4 Computations in the proof of Proposition 5.3

We set A and AI to be the matrices in Equation 5.6 and Equation 5.7.

```
A={{1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0}, {0,
    0, 0, w, 0, b x, -x}, {0, 0, 0, 0, y, -z, -a z}, {0, 0, 0, x, 0, -b w, w},
     {0, 0, 0, 0, z, y, a y}};
AI={{1, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0, 0}, {0,
    0, 0, w/a, 0, x/a, 0}, {0, 0, 0, 0, y/b, 0, z/b}, {0, 0, 0, x/(1 + a b),
    -(z/(b (1 + a b))), -(w/(1 + a b)), y/(b (1 + a b))}, {0, 0, 0, -(x/(a (1
    + a b))), -(z/(1 + a b)), w/(a (1 + a b)), y/(1 + a b)}};
```

Next we check $(A'_{a,b})^t S_F A'_{a,b} = S_{P'_{a,b}}$.

```
Simplify[Transpose[A].DiagonalMatrix[{-1, 1, 1, 1, 1, 1, 1}].A ==
    DiagonalMatrix[{-1, 1, 1, a, b, b (1 + a b), a (1 + a b)}], a == w^2 + x^2
     && b == y^2 + z^2]
True
```

A matrix N in $\Delta^{a,b}_{(2)}$ has form

$$\begin{pmatrix} 2a_1+1 & 2b_1 & 2c_1 & 2ad_1 & 2be_1 & 0 & 0 \\ 2a_2 & 2b_2+1 & 2c_2 & 2ad_2 & 2be_2 & 0 & 0 \\ 2a_3 & 2b_3 & 2c_3+1 & 2ad_3 & 2be_3 & 0 & 0 \\ 2a_4 & 2b_4 & 2c_4 & 2ad_4+1 & 2be_4 & 0 & 0 \\ 2a_5 & 2b_5 & 2c_5 & 2ad_5 & 2be_5+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We compute $A'_{a,b}N(A'_{a,b})^{-1}$.

```
Nmat = {{1 + 2 a1, 2 b1, 2 c1, 2 a d1, 2 b e1, 0, 0}, {2 a2, 1 + 2 b2, 2 c2,
    2 a d2, 2 b e2, 0, 0}, {2 a3, 2 b3, 1 + 2 c3, 2 a d3, 2 b e3, 0, 0}, {2 a4
    , 2 b4, 2 c4, 1 + 2 a d4, 2 b e4, 0, 0}, {2 a5, 2 b5, 2 c5, 2 a d5, 1 + 2
    b e5, 0, 0}, {0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 1}};

Expand[Simplify[A.Nmat.AI, a == w^2 + x^2 && b == y^2 + z^2]]
{{1 + 2 a1, 2 b1, 2 c1, 2 d1 w, 2 e1 y, 2 d1 x, 2 e1 z}, {2 a2, 1 + 2 b2, 2
    c2, 2 d2 w, 2 e2 y, 2 d2 x, 2 e2 z}, {2 a3, 2 b3, 1 + 2 c3, 2 d3 w, 2 e3 y
    , 2 d3 x, 2 e3 z}, {2 a4 w, 2 b4 w, 2 c4 w, 1 + 2 a d4 - 2 d4 x^2, 2 e4 w
    y, 2 d4 w x, 2 e4 w z}, {2 a5 y, 2 b5 y, 2 c5 y, 2 d5 w y, 1 + 2 b e5 - 2
    e5 z^2, 2 d5 x y, 2 e5 y z}, {2 a4 x, 2 b4 x, 2 c4 x, 2 d4 w x, 2 e4 x y,
    1 + 2 d4 x^2, 2 e4 x z}, {2 a5 z, 2 b5 z, 2 c5 z, 2 d5 w z, 2 e5 y z, 2 d5
     x z, 1 + 2 e5 z^2}}
```

Appendix B

Magma and Snap computations

This appendix contains computations done by the author using the programs Magma [BCP97] and Snap [CGHN00].

B.1 The figure-8 knot group

In this section we compute a special subgroup of the figure-8 knot group for Corollary 3.11 by applying the results in Theorem 3.1.

We compute the imagine of Γ_8 under reduction modulo 2, the dihedral group D_5:

```
> F:=QuadraticField(-3);
> R<w>:=RingOfIntegers(F);
> R2:=quo<R|2>;
> GL2:=GeneralLinearGroup(2,R2);
> x:=elt<GL2 | 1,1, 0,1>;
> y:=elt<GL2 | 1,0, w,1>;
> G<a,b>:=FPGroup(MatrixGroup<2,R2|x,y>);
> G;
Finitely presented group G on 2 generators
Relations
    a^2 = Id(G)
    b^2 = Id(G)
    (b * a)^5 = Id(G)
```

The group $\Gamma_8 \cap \mathrm{PSL}(2, \mathcal{O}_3)_{(2)}$ contains the normal closure of x^2 and y^2

and has index 10:

```
> G<x,y>:=Group<x,y| 1=x^-1*y^-1*x*y*x*y^-1*x^-1*y*x*y>;
> L:=LowIndexSubgroups(G,<10,10>: Subgroup:=ncl<G|x^2,y^2>);
> L;
[
    Finitely presented group on 8 generators
    Index in group G is 10 = 2 * 5
    Generators as words in group G
        $.1 = x^-2
        $.2 = y^-2
        $.3 = (x * y^-1 * x^-1)^2
        $.4 = (y * x^-1 * y^-1)^2
        $.5 = (x * y * x^-1 * y^-1 * x^-1)^2
        $.6 = (y * x * y^-1 * x^-1 * y^-1)^2
        $.7 = (x * y * x * y^-1 * x^-1 * y^-1 * x^-1)^2
        $.8 = (y * x * y * x^-1 * y^-1 * x^-1 * y^-1)^2
]
```

We then find its intersection with Δ_3, which we know has index 2, by looking for a subgroup containing certain elements which we can immediately check that their corresponding matrices are in Δ_3.

```
> K:=sub<L[1]|L[1].1,L[1].3,L[1].4,L[1].2^2,L[1].5*L[1].2>
> D:=LowIndexSubgroups(L[1],<2,2>: Subgroup:=K);
> D;
[
    Finitely presented group on 9 generators
    Index is 2
    Generators as words
        $.1 = $.1
        $.2 = $.3
        $.3 = $.4
        $.4 = $.2^2
        $.5 = $.5 * $.2
        $.6 = $.1
        $.7 = $.6 * $.2^-1
        $.8 = $.8
        $.9 = $.2 * $.1 * $.2^-1
```

```
]
> Simplify(D[1]);
Finitely presented group on 8 generators
Generators as words
    $.1 = $.1
    $.2 = $.2
    $.3 = $.3
    $.4 = $.4
    $.5 = $.5
    $.6 = $.7
    $.7 = $.8
    $.8 = $.9
Relations
    $.6 * $.2^-1 * $.6^-1 * $.3^-1 * $.1 * $.2 * $.1^-1 * $.3 = Id($)
    $.3^-1 * $.7^-1 * $.4^-1 * $.6^-1 * $.1^-1 * $.3 * $.1 * $.6 * $.4
        * $.7 = Id($)
    $.8^-1 * $.4 * $.5^-1 * $.7 * $.3^-1 * $.4^-1 * $.8 * $.4 * $.3 * $
        .7^-1 * $.5 * $.4^-1 = Id($)
    $.6^-1 * $.1^-1 * $.6 * $.2^-1 * $.6^-1 * $.3^-1 * $.4^-1 * $.1 * $
        .2 * $.6 * $.4 * $.7 * $.3 * $.7^-1 = Id($)
    $.6 * $.4 * $.7 * $.4^-1 * $.8^-1 * $.4 * $.3 * $.5 * $.4^-1 * $
        .6^-1 * $.3^-1 * $.4^-1 * $.8 * $.4 * $.7^-1 * $.5^-1 = Id($)
    $.2 * $.5 * $.7 * $.4^-1 * $.8^-1 * $.1 * $.2 * $.6 * $.4 * $.7 * $
        .3 * $.7^-1 * $.2^-1 * $.6^-1 * $.3^-1 * $.4^-1 * $.8 * $.4 * $
        .7^-1 * $.5^-1 * $.2^-1 * $.1^-1 = Id($)
    $.5^-1 * $.2^-1 * $.1^-1 * $.8 * $.4 * $.7 * $.4^-1 * $.8^-1 * $.1
        * $.2 * $.6 * $.4 * $.7 * $.3 * $.7^-1 * $.5 * $.4^-1 * $.8^-1
        * $.6^-1 * $.3^-1 * $.4^-1 * $.8 * $.4 * $.7^-1 = Id($)
```

B.2 The non-Haken example in Section 4.2

In this section we compute a special subgroup of a non-Haken cocompact Kleinian group by applying the results in Proposition 4.3.

We call F the maximal arithmetic group in $\mathrm{Isom}(\mathbb{H}^3)$ containing the group TR generated by the reflections on the side of the hyperbolic tetrahedron with dihedral angles $\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}; \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}\}$.

```
> F<a,b,c,d,f,g>:=Group<a,b,c,d,f,g| 1 = a^2 = b^2 = c^2 = d^2 = (a*b)
   ^3 = (a*c)^2 = (a*d)^4 = (b*c)^4 = (b*d)^2 = (c*d)^3 = f^2 = g^2 =
   f*a*f*d = f*b*f*c = g*a*g*b = g*c*g*d = g*f*g*f>;
> TR:=sub<F| a,b,c,d>;
```

The corresponding matrix elements are as follows:

$$a \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad b \leftrightarrow \begin{pmatrix} \frac{9}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -\frac{7}{2} & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ -\frac{21}{2} & -\frac{3}{2} & -\frac{7}{2} & -\frac{3}{2} \\ -\frac{7}{2} & -\frac{1}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix},$$

$$c \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad d \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$f \leftrightarrow \begin{pmatrix} \frac{3}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{7}{\sqrt{2}} & 0 & -\frac{3}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad g \leftrightarrow \begin{pmatrix} \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{7}{2} & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{7}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ -\frac{7}{2} & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

where a, b, c, d are orientation reversing isometries.

Let T be Γ, the index-2 subgroups of words of length 2 in TR. We call SO the group $\mathrm{SO}^+(Q'_7, \mathbb{Z})$ and D the group $\Gamma \cap \mathrm{SO}^+(Q'_7, \mathbb{Z})$.

```
> T:=sub<F | c*a, d*a, b*a>;
> SO:=sub<F | c*a, d*a, b*a*g>;
> D:=Simplify(T meet SO);
> D;
Generators as words in group F
    $.1 = c * a
    $.2 = d * a
    $.3 = b * a * g * c * a * g^-1 * a^-1 * b^-1
    $.4 = b * a * g * d * a * g^-1 * a^-1 * b^-1
Relations
    $.1^2 = Id($)
```

```
    $.3^2 = Id($)
    $.2^4 = Id($)
    ($.3 * $.2^-1)^2 = Id($)
    $.4^4 = Id($)
    ($.1 * $.4^-1)^2 = Id($)
    ($.1 * $.2^-1)^3 = Id($)
    ($.2^-1 * $.4)^3 = Id($)
    ($.3 * $.4^-1)^3 = Id($)
> Index(T,D);
3
> Index(SO,D);
2
```

Now, we know that the group Δ^7 has index 2 in the group $\mathrm{SO}^+(Q'_7, \mathbb{Z})$. Since the generators of the group D satisfy the conditions for membership in Δ^7, we can conclude D is Δ^7. Let D1$=ca$, D2$=da$, D3$=(bag)ca(bag)^{-1}$, and D3$=(bag)da(bag)^{-1}$.

The index $[\Delta^7 : \Delta^7_{(2)}]$ is equal to the order of the image after reduction modulo 2.

```
> GL:=GeneralLinearGroup(4,Integers(2));
> D1:=elt<GL | 1,0,0,0, 0,0,1,0, 0,1,0,0, 0,0,0,-1>;
> D2:=elt<GL | 1,0,0,0, 0,0,0,-1, 0,0,1,0, 0,1,0,0>;
> D3:=elt<GL | 8,0,3,0, 0,1,0,0, -21,0,-8,0, 0,0,0,-1>;
> D4:=elt<GL | 8,2,2,1, -14,-3,-4,2, -14,-4,-3,2, -7,-2,-2,0>;
> Order(MatrixGroup< 4, Integers(2) | D1,D2,D3,D4>);
24
```

The once-punctured torus bundle with monodromy R^2L^2 is manifold `m136` on the census. It has a non-Haken dehn filling (4,1). We read off arithmetic invariants of this non-Haken manifold on Snap.

```
1. : read manifold m136(4,1)
1. m136(4,1): print volume
```

```
Volume is: 2.666744783449059790796712463
1. m136(4,1): compute invariant_trace_field
Invariant trace field: x^2-x+2 [0,1] -7 R(1) = 0.500000000+1.32287566*I
1. m136(4,1): print group
< a b c | aacBCb acbbAC abABcccc >
M0= c L0= abAB
Original generators
a BC cbC C
[-0.69178+1.18518*i,-0.69178-0.22903*i; 0.61098+0.75386*i
    ,-0.69178-0.22903*i]
[ 1.49515+0.08805*i,-2.35921-2.15417*i; 0.17228-0.41195*i
    ,-0.17228+0.41195*i]
[-0.10998-0.51950*i,-0.95615+1.38355*i;-0.06612+0.86405*i,
    1.74222+0.11426*i]
1. m136(4,1): print arithmetic
Invariant trace field: x^2-x+2 [0,1] -7 R(1) = 0.500000000+1.32287566*I
Integer traces: YES
Invariant quaternion algebra: (4*x-12, 6*x-34)
Real ramification[]
Finite ramification:
[2,[1,1]~,1,1,[0,1]~]
[2,[4,1]~,1,1,[1,1]~]
Arithmetic: YES
Borel Regulator: not computed (shape field required)
1. m136(4,1): compute trace_field
Trace field: x^16-7*x^12+16*x^8-7*x^4+1 [0,8] 24759631762948096 R(1) =
    -1.36483992+0.142882769*I
```

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